

Supersymmetry and All That : A simple example in 1–dimension

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Outline :

- Very brief history
- N=1 SUSY in 1 dimension
 - Component formalism
 - Superfield formalism
- N=2 SUSY in 1 dimension
 - How to extend
 - An alternative way of obtaining the action

Supersymmetry

- 1967 Coleman-Mandula no-go theorem:

It is not possible to extend the Poincare group $(P_\mu, J_{\mu\nu})$ in a non-trivial way (i.e. the only way $:[P_\mu, \Omega] = 0 = [J_{\mu\nu}, \Omega]$)

- 1971 Golfand-Likhtman - Birth of SUSY

If a Lie group has a graded structure it is possible to extend the Poincare group. \Rightarrow Superalgebra (Z_2 graded structure)

$$SUSY : Fermion \rightarrow Boson, Boson \rightarrow Fermion$$

- 1970's Superstring theories

- 1974 Wess-Zumino Model

First renormalizable theory in 4-dim.s \Rightarrow SUSY becomes popular

- 1975 Haag-Lopusanski-Sohnius

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- 201? LHC \Rightarrow ???

Supersymmetric theories are highly restricted :

- Bosons and fermions can only be related to each other by fermionic symmetry operators Q of spin $-1/2$ (not spin $-3/2$ or higher).

$$Q|fermion \rangle = |boson \rangle \quad , \quad Q|boson \rangle = |fermion \rangle$$

- Only in the presence of SUSY, multiplets can contain particles of different spin.
- Particles in the same supermultiplet have the same mass and coupling constant.
- no. of bosons = no. of fermions in a supersymmetric theory.

One can write SUSY transformations and supersymmetric actions in two different ways :

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- Write the most general action that we know from field theory including kinetic, mass, interaction terms.
- Fix the coefficients in the action so that it is invariant under SUSY.
- This is a tedious but a straightforward way to construct.
(See for instance book by West for details.)

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- Supersymmetric actions can then be written directly in terms of **SUPERFIELDS** $\Phi(x, \theta, \bar{\theta})$ and their super derivatives.
- This is an elegant way to construct but may not work for every case.
(See any book on SUSY for details.)

A Simple Example in 1-Dimension :

Let us consider,

- A real scalar field $\phi(t)$,
- A real fermionic field $\psi(t)$.

For each t , $\psi(t)$, is an independent Grassmann variable :

$$\psi(t_1)\psi(t_2) = -\psi(t_2)\psi(t_1) \Rightarrow (\psi(t))^2 = 0$$

Assume that $d\psi(t)/dt \equiv \dot{\psi}(t)$ exists then we also have

$$\dot{\psi}(t_1)\dot{\psi}(t_2) = -\dot{\psi}(t_2)\dot{\psi}(t_1), \dot{\psi}(t)\psi(t) = -\psi(t)\dot{\psi}(t)$$

Therefore ψ and $\dot{\psi}$ anticommute with themselves and with each other at equal time t :

$$\{\psi(t), \psi(t)\} = 0, \{\psi(t), \dot{\psi}(t)\} = 0, \{\dot{\psi}(t), \dot{\psi}(t)\} = 0$$

Note the difference between ϕ and ψ

$$\int dt \phi(t) \dot{\phi}(t) = \frac{1}{2} \int dt \frac{d}{dt} (\phi(t) \phi(t))$$

but

$$\int dt \psi(t) \dot{\psi}(t) \neq \frac{1}{2} \int dt \frac{d}{dt} (\psi(t) \psi(t))$$

As an action for these fields we take,

$$I = \int dt \left(\frac{1}{2} \dot{\phi}^2 + \frac{i}{2} \psi \dot{\psi} \right).$$

Note that in this action,

- $\frac{1}{2} \dot{\phi}^2$ term is a truncation of the Klein-Gordon action to an (x, y, z) independent field,
- $\frac{i}{2} \psi \dot{\psi}$ term is a truncation of the Dirac action for a real spinor to one of its component that is also independent of (x, y, z) .

For $\hbar = 1$ the dimension of the action is zero ($[I] = 0$), therefore taking $[t] = -1$ we get the dimensions of the fields as

$$[\phi] = -\frac{1}{2}, [\psi] = 0$$

SUSY Transformations

We look for a symmetry such that

$$\begin{aligned}\delta_{\xi}(\textit{boson}) &= \xi(\textit{somethingfermionic}) \\ \delta_{\xi}(\textit{fermion}) &= \xi(\textit{somethingbosonic})\end{aligned}$$

It is clear that ξ must be anticommuting !

- Let $\delta_{\xi}\phi = i\xi\psi$. Then $[\xi] = -1/2$.

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Therefore, let us consider $\delta_{\xi}\psi = \xi f(\phi, \dot{\phi})$ such that $[f] = 1/2$.
- If the transformation is linear the only possible choice is $f \sim \xi\dot{\phi}$.

Indeed, in order to get $\delta_{\xi}I = 0$ we find

$$\delta_{\xi}\psi = -\xi\dot{\phi}.$$

Summary :

- SUSY transformations : $\delta_\xi \phi = i\xi\psi$, $\delta_\xi \psi = -\xi\dot{\phi}$.
- $I = \int dt \left(\frac{1}{2}\dot{\phi}^2 + \frac{i}{2}\psi\dot{\psi} \right)$ is invariant (i.e. supersymmetric).
- (ϕ, ψ) real scalar supermultiplet.
- # of bosons = # of fermions

SUSY Algebra :

To get the algebra let us study the commutator of two SUSY transformations:

$$[\delta_\eta, \delta_\xi]\phi = (\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\phi = 2i\eta\xi\dot{\phi}$$

$$[\delta_\eta, \delta_\xi]\psi = 2i\eta\xi\dot{\psi}$$

We get,

$$[\delta_\eta, \delta_\xi] = 2i\eta\xi \left(\frac{d}{dt} \right)$$

for constant parameters ξ, η .

Note that right hand side is a translation over a distance $t_0 = 2i\eta\xi$!

Let us obtain the algebra in a tricky way. We have

$$i \frac{d}{dt} \equiv H, \quad \delta_\xi \equiv i\xi Q, \quad \delta_\eta \equiv i\eta Q$$

and H ve Q denotes the generators of translation and SUSY. The commutator of SUSY transformations can be written in terms of anticommutators as

$$[\delta_\eta, \delta_\xi] = -(\xi Q \eta Q - \eta Q \xi Q) = \xi \eta (Q Q + Q Q) = \xi \eta \{Q, Q\}$$

With the help of above definitions we get

$$\{Q, Q\} = 2H$$

Jacobi identity gives the other relation :

$$[Q, H] = 0$$

Therefore, SUSY algebra in one dimension is

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- Note that, algebra contains both commutators and anticommutators,
- therefore, Q and H generators form a graded Lie algebra as promised before.

REMARK :

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- 1970's $\sqrt{H} \sim Q$: Supersymmetry
 \Rightarrow prediction : for every particle there should be a superpartner .
(201?, will LHC find one?)

Grassmann Algebra :

Let $\theta_i, i = 1, 2 \dots n$ to be n Grassmann numbers that satisfies,

$$\theta_i \theta_j + \theta_j \theta_i = 0, \forall i, j \rightarrow \theta_i \theta_i = 0$$

Definition of derivative and integration is given as

$$\frac{\partial}{\partial \theta_i} \theta_j = \delta_{ij}, \int d\theta_1 d\theta_2 \dots d\theta_n \theta_n \dots \theta_2 \theta_1 = 1$$

Note that integral operates as a derivative!

The above relations simplify a lot for one θ :

$$\int d\theta \theta = 1, \int d\theta c = 0 \rightarrow \int d\theta \equiv \frac{d}{d\theta}$$

Superspace :

Since in one dimension we have one Q and H we parametrize the space with θ and t where it is obvious that both are real and θ is Grassman variable.

DEFINITION :

- The space with coordinates t and θ is called SUPERSPACE.
- Any function of t and θ (i.e. $\Phi(t, \theta)$) is called a SUPERFIELD.

Then since, $\theta^2 = 0$ in our model the simplest superfield is,

$$\Phi(t, \theta) = \phi(t) + i\theta\psi(t)$$

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- All components of Φ must be scalar. Therefore, since θ is a Grassmann variable, ψ must be a anticommuting field.
- All components of Φ must be real.
That's why we have an i in front $\theta\psi$.
- Since $[\Phi] = [\phi] = -1/2$ we must also have $[\psi] = 0$ and $[\theta] = -1/2$.

SUSY transformations can be obtained with the help of a Hermitian operator,

$$Q = \frac{\partial}{\partial \theta} + i\theta \frac{\partial}{\partial t}$$

because

$$\xi Q \Phi = \xi \left(\frac{\partial}{\partial \theta} + i\theta \frac{\partial}{\partial t} \right) \Phi = i\xi \psi + i\theta(-\xi \dot{\phi}) = \delta \phi + i\theta \delta \psi$$

Note that this operator Q satisfies SUSY algebra :

$$\{Q, Q\} = 2i \frac{\partial}{\partial t} \equiv H, [Q, H] = 0, [H, H] = 0$$

The invariance of the action can be written as

$$\delta I = \int dt d\theta \xi Q[\dots] = 0.$$

It is important to know other operators that commute with ξQ . One of them is d/dt 'dir. The other one is defined as

$$D = \frac{\partial}{\partial \theta} - i\theta \frac{\partial}{\partial t}$$

and it is called super covariant derivative.

From the definition we see that,

$$[\xi Q, D] = \xi\{Q, D\} = 0.$$

Let us construct the action by using "*what else can it be*" method. In general we can write,

$$I = \int dt d\theta F\left(\frac{\partial}{\partial t}, D, \Phi\right).$$

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- A physically interesting action must at least be quadratic in fields. So we get

$$F = K\left(\frac{\partial}{\partial t}, D\right) \cdot \Phi^2.$$

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- A physically interesting action must at least be quadratic in fields. So we get

$$F = K\left(\frac{\partial}{\partial t}, D\right) \cdot \Phi^2.$$

- Since $[\Phi] = -1/2$ we can only have $[K] = 3/2$. So only by doing dimensional analysis we get only one solution for K :

$$K \sim \frac{\partial}{\partial t} \cdot D.$$

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- If we want at most 2nd derivatives of the fields we have an unique solution :

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- which gives

$$\begin{aligned} I &= \frac{i}{2} \int dt d\theta (\dot{\phi} + i\theta\dot{\psi}) (i\psi - i\theta\dot{\phi}) = 0 + \frac{i}{2} \int dt d\theta (-i\theta\dot{\phi}\dot{\phi} + \theta\psi\dot{\psi}) + 0 \\ &= \int dt d\theta \left(\frac{1}{2}\theta\dot{\phi}\dot{\phi} + \frac{i}{2}\theta\psi\dot{\psi} \right) \end{aligned}$$

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- This is the same action that we obtained before without using superspace techniques !

Extended SUSY :

Now let us consider more than one SUSY in one dimension, i.e. we have N SUSY generators Q^i , $i = 1, 2, \dots, N$ so that

$$\delta_\xi \phi = \xi_i Q_i \phi = i \sum_{i=1}^N \xi_i \psi_i \quad , \quad \delta_\xi \psi = \xi_i Q_i \psi = -\xi_i \dot{\phi}.$$

The action,

$$I = \int dt \left(\frac{1}{2} \dot{\phi} \dot{\phi} - \frac{i}{2} \sum_{i=1}^N \psi_i \dot{\psi}_i \right)$$

is still invariant under this extended SUSY transformations.

But there is something unusual

There are N fermions but still 1 boson !

Let us consider N=2 and check SUSY algebra,

$$[\delta_\xi, \delta_\eta]\phi = 2i(\xi_1\eta_1 + \xi_2\eta_2)\dot{\phi}$$

as expected but for instance for ψ_1 we get

$$[\delta_\xi, \delta_\eta]\psi_1 = 2i\xi_1\eta_1\dot{\psi}_1 + i(\xi_1\eta_2 + \xi_2\eta_1)\dot{\psi}_2$$

and it doesn't close on translation of ψ_1 unless we use equation of motion of ψ_2 : $\dot{\psi}_2 = 0$.

Such a SUSY called *onshell SUSY*

Since we have 2 fermions ψ_1, ψ_2 and one boson ϕ , and algebra doesn't close automatically to cure the problem let us introduce another boson field F such that

$$I = \int dt \left(\frac{1}{2} \dot{\phi} \dot{\phi} - \frac{i}{2} \psi_i \dot{\psi}_i + \frac{1}{2} F^2 \right)$$

Note that F doesn't have a kinetic term and it is called *auxiliary field*.

One can view (ϕ, ψ_1) as one multiplet and (ψ_2, F) as another such that

$$\delta\phi = i\xi_1\psi_1, \quad \delta\psi_1 = -\xi_1\dot{\phi}, \quad \delta\psi_2 = \xi_2 F, \quad \delta F = i\xi_2\dot{\psi}_2$$

with

$$I_1 = \int dt \left(\frac{1}{2} \dot{\phi} \dot{\phi} - \frac{i}{2} \psi_1 \dot{\psi}_1 \right), \quad I_2 = \int dt \left(-\frac{i}{2} \psi_2 \dot{\psi}_2 + \frac{1}{2} F^2 \right)$$

i.e. $N = 2 = 1 + 1$

However, one can construct N=2 SUSY as N=2 ! (Like for the 4-dimensional case.)

Let us write more general SUSY transformations by analyzing dimensions of the fields.

$$\delta\phi = i\xi_i\psi_i \quad , \quad \delta\psi_i = \xi_i\dot{\phi} + \alpha_{ij}\xi_j F \quad , \quad \delta F = i\xi_i\beta_{ij}\dot{\psi}_j$$

where α and β are real matrices.

From the commutator algebra

$$[\delta_\xi, \delta_\eta]\phi = 2i\eta_i\xi_i\dot{\phi} + (i\eta_i(\alpha_{ij} + \alpha_{ji})\xi_j F)$$

$$[\delta_\xi, \delta_\eta]\psi_i = i(\eta_i\xi_j - \xi_i\eta_j)\dot{\psi}_j + i\alpha_{ij}\beta_{kl}(\eta_j\xi_k - \xi_j\eta_k)\dot{\psi}_l$$

we get

- $\alpha_{ij} + \alpha_{ji} = 0 \quad , \quad \alpha_{ij}\beta_{jk} = \delta_{ik}$

so that $[\delta_\xi, \delta_\eta]$ closes to translations.

Finally, off-shell N=2 SUSY algebra in 1-dimension can be written as,

$$\delta\phi = i\xi_i\psi_i \quad , \quad \delta\psi_i = \xi_i\dot{\phi} + \epsilon_{ij}\xi_j F \quad , \quad \delta F = \xi_i\epsilon_{ij}\dot{\psi}_j$$

where $\epsilon^{12} = -\epsilon^{21} = 1$.

Moreover, since we have two fermions now we can write mass term

$$I_m = -m \int dt (F\phi + i\psi_1\psi_2)$$

and an interaction term

$$I_g = g \int dt \left(\frac{1}{2} F\phi^2 + i\psi_1\psi_2\phi \right)$$

An alternative way to obtain the action :

Note that

- Fields they belong to a supersymmetric multiplet.
- One can move from the lowest member of the multiplet to highest one by SUSY transformation.
- Since Action is supersymmetric it also belongs to a SUSY multiplet.
- Therefore, one should be able to obtain the action by applying multiple SUSY variations to a lower dimensional integrated field polynomial.

Of course, one can say that this observation is related with superspace.
But,

- we need off-shell formulation to write superfields,
- and off-shell formulation does not always exist !

Let us show, how this method works for our simple model : We can write the SUSY transformation as,

$$\delta_\xi = \xi_1 Q_1 + \xi_2 Q_2$$

For simplicity let us also define

$$Q = \frac{Q_1 + iQ_2}{\sqrt{2}} \quad , \quad \bar{Q} = \frac{Q_1 - iQ_2}{\sqrt{2}} \quad , \quad \psi = \frac{\psi_1 + i\psi_2}{\sqrt{2}} \quad , \quad \bar{\psi} = \frac{\psi_1 - i\psi_2}{\sqrt{2}}$$

We can write Q and \bar{Q} variations as

$$Q\phi = i\psi, \quad , \quad Q\psi = 0 \quad , \quad Q\bar{\psi} = \dot{\phi} - iF \quad , \quad QF = \dot{\psi}$$

$$\bar{Q}\phi = i\bar{\psi}, \quad , \quad \bar{Q}\bar{\psi} = 0 \quad , \quad \bar{Q}\psi = \dot{\phi} + iF \quad , \quad \bar{Q}F = -\dot{\bar{\psi}}$$

Then by analysis the dimensions of the fields and parameters,

$$[t] = -1, [m] = 1, [g] = \frac{3}{2}, [Q] = \frac{1}{2}, [\phi] = -\frac{1}{2}, [\psi] = 0, [F] = \frac{1}{2}$$

it is easy to get,

$$I = \int dt \left[-(\bar{Q}Q)^2 \phi^2 + \bar{Q}Q \left(\frac{m}{2} \phi^2 + \frac{g}{3!} \phi^3 \right) \right]$$

In other words, we can write the action by applying multiple super variations of the monomials of ϕ .

This also true for the on-shell transformations except that one gets the action modulo equation of motion of fermion fields!

A similar construction also works in 4 dimensions for WZ, N=1 and N=2 SYM (K.U, MPLA'XX) and even in much more complicated cases (H.Sonoda, K.U,2009).

Reference

- Nearly all of this talk is from the excellent lectures :
P. van Nieuwenhuizen "Supersymmetry, Supergravity, Superspace and BRST Symmetry in a Simple Model " arXiv: hep-th/0408179
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J. Wess and J. Bagger, "Supersymmetry and Supergravity", (1992).