On Integrable Systems in Cosmology

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- Dynamics on Lie groups
- Geodesic flow on $GL(n, \mathbb{R})$ group manifold
- Homogeneous cosmological models

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More than thirty five years after the famous paper of

Francesco Calogero

Solution of the one-dimensional n-body problem with quadratic and/or inversely quadratic pair potentials

Journal of Mathematical Physics 12 (1971) 419-436

Bill Sutherland

Physical Review A 1971, 1972

Jürgen Moser

Adv. Math. 1975

The **Calogero-Sutherland-Moser** type system which consists of n-particles on a line interacting with pairwise potential $V(x)$ admits the following spin generalization

$$H_{ECM} = \frac{1}{2}\sum_{a=1}^{n} p_a^2 + \frac{1}{2}\sum_{a\neq b}^{n} f_{ab} f_{ba} V(x_a - x_b)$$

The **Euler-Calogero-Sutherland-Moser** type systems are integrable in the following cases

I.	$\mathbf{V(z)} = z^{-2}$	**Calogero**
II.	$\mathbf{V(z)} = a^2 \sinh^{-2} az$	
III.	$\mathbf{V(z)} = a^2 \sin^{-2} az$	**Sutherland**
IV.	$\mathbf{V(z)} = a^2 \wp(az)$	
V.	$\mathbf{V(z)} = z^{-2} + \omega z^2$	**Calogero**

The nonvanishing Poisson bracket relations

$$\{x_i, p_j\} = \delta_{ij} \,, \qquad \{f_{ab}, f_{cd}\} = \delta_{bc} f_{ad} - \delta_{ad} f_{cb}$$

Dynamics on Lie groups

Let G be a real Lie group and $A(G)$ be its Lie algebra. The action of the Lie group G in its group manifold is given by the following diffeomorphisms

$$L_g : G \longrightarrow G \qquad L_g h = gh$$
$$R_g : G \longrightarrow G \qquad R_g h = hg \qquad h, g \in G$$

The adjoint and coadjoint actions are

$$Ad_g : A(G) \longrightarrow A(G) \quad Ad_{gh} = Ad_g Ad_h$$
$$Ad_g^* : A(G) \longrightarrow A(G) \quad Ad_{gh}^* = Ad_h^* Ad_g^*$$

The Lie-Poisson brackets are

$$\{F, H\} =< x, [\nabla F(x), \nabla H(x)] > ,$$
$$\{F, H\} = C^k_{ij} x_k \frac{\partial F}{\partial x_i} \frac{\partial H}{\partial x_j}, \quad x \in A^*(G)$$

The left and right invariant 1-forms obey the First structure Cartan equation

$$d\omega^L + \omega^L \wedge \omega^L = 0 ,$$
$$d\omega^R - \omega^R \wedge \omega^R = 0$$

The geodesic flow on the matrix Lie group G is described by the map

$$t \longmapsto g(t) \in G$$

and by the Lagrangian

$$L = \frac{1}{2} < \omega, \omega >, \qquad \omega \in A(G)$$

In local coordinates we have

$$L = \frac{1}{2} G_{\mu\nu} X^{\mu} X^{\nu}$$

The phase space is the cotangent bundle $T^*(G)$ with canonical symplectic structure

$$\Omega = d(P_\mu dX^\mu)$$

$$\{X^\mu, X^\nu\} = 0\,, \{P_\mu, P_\nu\} = 0\,, \{X^\mu, P_\nu\} = \delta^\mu_\nu$$

and Hamiltonian

$$H = \frac{1}{2} G^{\mu\nu} P_\mu P_\nu$$

Let us define the functions

$$\xi_\mu^{L,R} = \Omega^{L,R\ \nu}{}_\mu P_\nu$$

such that

$$\xi_\mu^R = Ad_\mu^\nu \xi_\nu^L$$

with Poisson bracket relations

$$\{\xi_\alpha^R, \xi_\beta^R\} = -C_{\alpha\beta}^\gamma \xi_\gamma^R\,,$$
$$\{\xi_\alpha^L, \xi_\beta^L\} = C_{\alpha\beta}^\gamma \xi_\gamma^R\,,$$
$$\{\xi_\alpha^R, \xi_\beta^R\} = 0$$

Geodesic flow on $GL(n,\mathbb{R})$ group manifold

Let us consider the Lagrangian

$$L = \frac{1}{2} < \omega, \omega >, \quad \omega \in gl(n,\mathbb{R})$$

which is defined by the following one parameter family of non-singular metrics on $GL(n,\mathbb{R})$ group manifold

$$< X\,, Y >= \mathbf{Tr} XY - \frac{\alpha}{\alpha n - 2} \mathbf{Tr} X\, \mathbf{Tr} Y\,,$$

where $X, Y \in gl(n,\mathbb{R})$ and $\alpha \in [0,1]$

Thus we have

$$L = \frac{1}{2}\,\mathbf{tr}\left(g^{-1}\dot{g}\right)^2 - \frac{\alpha}{2(\alpha n - 2)}\,\mathbf{tr}^2\left(g^{-1}\dot{g}\right) =$$

$$= \frac{1}{2}\,G^{-1}_{ab,cd}\,\dot{g}_{ab}\,\dot{g}_{cd}$$

Here $g \in GL(n,\mathbb{R})$ **and**

$$G_{ab,cd} = g_{ad}\,g_{cb} - \frac{\alpha}{2}\,g_{ab}\,g_{cd}\,,$$

$$G^{-1}_{ab,cd} = g^{-1}_{da}\,g^{-1}_{bc} - \frac{\alpha}{\alpha n - 2}\,g^{-1}_{ba}\,g^{-1}_{dc}$$

The invariance of the Lagrangian under the left and right translations leads to the possibility of explicit integration of the dynamical equations

$$\frac{d}{dt}\left(g^{-1}\dot{g}\right) = 0 \Rightarrow g(t) = g(0)\exp\left(tJ\right)$$

The canonical Hamiltonian corresponding to the bi-invariant Lagrangian L is

$$H = \frac{1}{2}\,\mathbf{tr}\left(\pi^T g\right)^2 - \frac{\alpha}{4}\,\mathbf{tr}^2\left(\pi^T g\right) = \frac{1}{2}\,G_{ab,cd}\,\pi_{ab}\,\pi_{cd}$$

The nonvanishing Poisson brackets between the fundamental phase space variables are

$$\{g_{ab}\,, \pi_{cd}\} = \delta_{ab}\,\delta_{cd}$$

At first we would like to analyze the following symmetry action on $SO(n,\mathbb{R})$ **in** $GL(n,\mathbb{R})$

$$g \mapsto g\prime = R\,g$$

It is convenient to use the polar decomposition for an arbitrary element of $GL(n,\mathbb{R})$

$$g = OS$$

The orthogonal matrix $O(\phi)$ is parameterized by the Euler angles $(\phi_1, \cdots, \phi_{\frac{n(n-1)}{2}})$ and S is a positive definite symmetric matrix

We can treat the polar decomposition as $1-1$ transformation from n^2 variables g to a new set of Lagrangian variables:

$$g \longmapsto (S_{ab}\,, \phi_a)\ ,$$

the $\frac{n(n-1)}{2}$ Euler angles and $\frac{n(n+1)}{2}$ symmetric matrix variables S_{ab}

In terms of these new variables the Lagrangian can be rewritten as

$$L_{GL} = \frac{1}{2}\,\mathbf{tr}\left(\Theta_L + \dot{S}\,S^{-1}\right)^2$$

The polar decomposition induces a canonical transformation

$$(g,\pi) \longmapsto (S_{ab}, P_{ab}; \phi_a, P_a)$$

The canonical pairs obey the Poisson bracket relations

$$\{S_{ab}\,, P_{cd}\} = \frac{1}{2}\,(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\;, \qquad \{\phi_a\,, P_b\} = \delta_{ab}$$

In the case of $GL(3,\mathbb{R})$ the orthogonal matrix is given by $O(\phi_1,\phi_2,\phi_3) = e^{\phi_1 J_3}\, e^{\phi_2 J_1}\, e^{\phi_3 J_3} \in SO(3,\mathbb{R})$. The canonical transformation is given by

$$(g,\pi) \longmapsto (S_{ab}, P_{ab}; \phi_a, P_a)\ ,$$

where

$$\pi = O\left(P - k_a J_a\right)\ ,$$
$$k_a = \gamma^{-1}_{ab}\left(\eta^L_b - \varepsilon_{bmn}\left(SP\right)_{mn}\right)$$

and

$$\gamma_{ik} = S_{ik} - \delta_{ik}\ \mathbf{tr}S$$

Here η_a^L are three left-invariant vector fields on $SO(3,\mathbb{R})$

$$\eta_1^L = -\frac{\sin\phi_3}{\sin\phi_2}\,P_1 - \cos\phi_3\,P_2 + \cot\phi_2\sin\phi_3\ P_3\,,$$
$$\eta_2^L = -\frac{\cos\phi_3}{\sin\phi_2}\,P_1 + \sin\phi_3\,P_2 + \cot\phi_2\cos\phi_3\ P_3\,,$$
$$\eta_3^L = -P_3$$

The right-invariant vector fields are

$$\eta_1^R = -\sin\phi_1\ \cot\phi_2 P_1 + \cos\phi_1\ P_2 + \frac{\sin\phi_1}{\sin\phi_2}\ P_3\,,$$

$$\eta_2^R = \cos\phi_1\ \cot\phi_2\ P_1 + \sin\phi_1\ P_2 - \frac{\cos\phi_1}{\sin\phi_2} P_3\,,$$

$$\eta_3^R = P_1$$

In terms of the new variables the canonical Hamiltonian takes the form

$$H = \frac{1}{2}\mathbf{tr}\,(PS)^2 - \frac{\alpha}{4}\,\mathbf{tr}^2\,(PS) + \frac{1}{2}\mathbf{tr}\,(J_a S J_b S)\,k_a k_b$$

The canonical variables $(S_{ab}\,, P_{ab})$ are invariant while $(\phi_a\,, P_a)$ undergo changes

The configuration space $\mathcal{S}$ of the real symmetric 3×3 matrices can be endowed with the flat Riemannian metric

$$ds^2 =< dQ\,, dQ >= \mathbf{Tr}\, dQ^2\,,$$

whose group of isometry is formed by orthogonal transformations $Q' = R^T\, Q\, R$. The system is invariant under the orthogonal transformations $S' = R^T\, S\, R$. The orbit space is given as a quotient space $\mathcal{S}/SO(3, \mathbb{R})$ which is a stratified manifold

(1) *Principal orbit-type stratum*,
when all eigenvalues are unequal $x_1 < x_2 < x_3$
with the smallest isotropy group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$

(2) *Singular orbit-type strata*
forming the boundaries of orbit space with

(a) two coinciding eigenvalues (e.g. $x_1 = x_2$) ,
when the isotropy group is $SO(2) \otimes \mathbb{Z}_2$

(b) all three eigenvalues are equal $(x_1 = x_2 = x_3)$,
here the isotropy group coincides with the isometry group $SO(3, \mathbb{R})$

Hamiltonian on the Principal orbit

To write down the Hamiltonian describing the motion on the principal orbit stratum we use the main-axes decomposition in the form

$$S = R^T(\chi) e^{2X} R(\chi) \,,$$

where $R(\chi) \in SO(3, \mathbb{R})$ is parametrized by three Euler angles $\chi = (\chi_1, \chi_2, \chi_3)$ and e^{2X} is a diagonal $e^{2X} = \textbf{diag} \, \| e^{2x_1}, e^{2x_2}, e^{2x_3} \|$

The original physical momenta P_{ab} are expressed in terms of the new canonical pairs

$$(S_{ab}, P_{ab}) \longmapsto (x_a, p_a; \chi_a, P_{\chi_a}) \ ,$$

where

$$P = R^T e^{-X} \left(\sum_{a=1}^{3} \bar{\mathcal{P}}_a \bar{\alpha}_a + \sum_{a=1}^{3} \mathcal{P}_a \alpha_a \right) e^{-X} R$$

with

$$\bar{\mathcal{P}}_a = \frac{1}{2}\, p_a \,,$$
$$\mathcal{P}_a = -\frac{1}{4}\frac{\xi_a^R}{\sinh(x_b - x_c)}, \qquad (\textbf{cyclic}\;\; a \neq b \neq c)$$

In the representation we introduce the orthogonal basis for the symmetric 3×3 matrices $\alpha_A = (\overline{\alpha}_i,\ \alpha_i)$, $i = 1, 2, 3$ with the scalar product

$$\mathbf{tr}(\bar{\alpha}_a\,\bar{\alpha}_b) = \delta_{ab}\,, \quad \mathbf{tr}(\alpha_a\,\alpha_b) = 2\delta_{ab}\,, \quad \mathbf{tr}(\bar{\alpha}_a\,\alpha_b) = 0$$

In this case the $SO(3,\mathbb{R})$ right-invariant Killing vector fields are

$$\xi_1^R = -p_{\chi_1} \, ,$$

$$\xi_2^R = \sin\chi_1 \cot\chi_2 \; p_{\chi_1} - \cos\chi_1 \; p_{\chi_2} - \frac{\sin\chi_1}{\sin\chi_2} \; p_{\chi_3} \, ,$$

$$\xi_3^R = - \; \cos\chi_1 \cot\chi_2 \; p_{\chi_1} - \sin\chi_1 \; p_{\chi_2} + \frac{\cos\chi_1}{\sin\chi_2} \; p_{\chi_3}$$

After passing to these main-axes variables the canonical Hamiltonian reads

$$H = \frac{1}{8}\sum_{a=1}^{3} p_a^2 - \frac{\alpha}{16}\left(\sum_{a=1}^{3} p_a\right)^2$$

$$+\frac{1}{16}\sum_{(abc)} \frac{(\xi_a^R)^2}{\sinh^2(x_b - x_c)} - \frac{1}{4}\sum_{(abc)} \frac{\left(R_{ab}\eta_b^L + \frac{1}{2}\xi_a^R\right)^2}{\cosh^2(x_b - x_c)}$$

The integrable dynamical system describing the free motion on principal orbits represents the **Generalized Euler-Calogero-Moser-Sutherland** model

Reduction to Pseudo-Euclidean Euler-Calogero-Moser model

We demonstrate how **IIA_2 Euler-Calogero-Moser-Sutherland** type model arises from the Hamiltonian after projection onto a certain invariant submanifold determined by discrete symmetries. Let us impose the condition of symmetry of the matrices $g \in GL(n, \mathbb{R})$

$$\psi_a^{(1)} = \varepsilon_{abc} g_{bc} = 0$$

One can check that the invariant submanifold of $T^*(GL(n, \mathbb{R}))$ is defined by

$$\Psi_A = (\psi_a^{(1)}, \psi_a^{(2)})$$

and the dynamics of the corresponding induced system is governed by the reduced Hamiltonian

$$H|_{\Psi_A=0} = \frac{1}{2}\,\mathbf{tr}\,(PS)^2 - \frac{\alpha}{4}\,\mathbf{tr}^2\,(PS)$$

The matrices S and P are now symmetric and nondegenerate.

It can be shown that

$$H_{PECM} = \sum_{a=1}^{3} p_a^2 - \frac{\alpha}{2}\left(p_1 + p_2 + p_3\right)^2 + \frac{1}{2}\sum_{(abc)} \frac{(\xi_a^R)^2}{\sinh^2(x_b - x_c)}$$

which is a **Pseudo-Euclidean version of the Euler-Calogero-Moser-Sutherland** model.

After the projection the Poisson structure is changed according to the Dirac prescription

$$\{F, G\}_D = \{F, G\}_{PB} - \{F, \psi_A\} C^{-1}_{AB} \{\psi_B, G\}$$

To verify this statement it is necessary to note that the Poisson matrix $C_{AB} = \|\{\psi^{(1)}_a, \psi^{(2)}_b\}\|$ is not degenerate. The resulting fundamental Dirac brackets between the main-axes variables are

$$\{x_a, p_b\}_D = \frac{1}{2}\,\delta_{ab}\,, \qquad \{\chi_a, p_{\chi_b}\}_D = \frac{1}{2}\,\delta_{ab}$$

The Dirac bracket algebra for the right-invariant vector fields on $SO(3,\mathbb{R})$ reduces to

$$\{\xi_a^R, \xi_b^R\}_D = \frac{1}{2}\,\varepsilon_{abc}\,\xi_c^R\,.$$

All angular variables are gathered in the Hamiltonian in three left-invariant vector fields η_a^L. The corresponding right-invariant fields $\eta_a^R = O_{ab}\,\eta_b^L$ are the integrals of motion

$$\{\eta_a^R, H_{PECM}\} = 0$$

Thus the surface on the phase space, determined by the constraints

$$\eta_a^R = 0$$

defines the invariant submanifold. Using the relation between left and right-invariant Killing fields $\eta_a^R = O_{ab}\,\eta_b^L$ we find out that after projection to the constraint surface the Hamiltonian reduces to

$$4\,H_{PECM} = \sum_a^3 p_a^2 - \frac{\alpha}{2}\,(p_1+p_2+p_3)^2 + \sum_{(abc)} \frac{(\xi_a^R)^2}{\sinh^2 2(x_b - x_c)}$$

Apart from the integrals η^R the system possesses other integrals. The integrals of motion corresponding to the geodesic motion with respect to the bi-invariant metric on $GL(n,\mathbb{R})$ group are

$$J_{ab} = (\pi^T g)_{ab}$$

The algebra of this integrals realizes on the symplectic level the $GL(n,\mathbb{R})$ algebra

$$\{J_{ab}, J_{cd}\} = \delta_{bc} J_{ad} - \delta_{ad} J_{cb}$$

After transformation to the scalar and rotational variables the expressions for the current J reads

$$J = \frac{1}{2}\sum_{a=1}^{3} R^T \left(p_a\,\bar{\alpha}_a - i_a\,\alpha_a - j_a J_a\right) R\,,$$

where

$$i_a = \sum_{(abc)} \frac{1}{2}\,\xi^R_a\,\coth(x_b - x_c) + \left(R_{am}\,\eta^L_m + \frac{1}{2}\,\xi^R_a\right)\tanh(x_b - x_c)$$

and

$$j_a = R_{am}\,\eta^L_m + \xi^R_a$$

After performing the reduction to the surface defined by the vanishing integrals

$$j_a = 0$$

we again arrive at the same Pseudo-Euclidean version of Euler-Calogero-Moser-Sutherland system.

Lax representation for the Pseudo-Euclidean Euler-Calogero-Moser model

The Hamiltonian equations of motion can be written in a Lax form

$$\dot{L} = [A, L]\,,$$

where the 3×3 matrices are given explicitly as

$$L = \begin{pmatrix} p_1 - \frac{\alpha}{2}(p_1+p_2+p_3) & L_3^+, & L_2^- \\ L_3^-, & p_2 - \frac{\alpha}{2}(p_1+p_2+p_3) & L_1^+ \\ L_2^+, & L_1^-, & p_3 - \frac{\alpha}{2}(p_1+p_2+p_3) \end{pmatrix}$$

$$A = \frac{1}{4}\begin{pmatrix} 0 & -A_3, & A_2 \\ A_3, & 0, & -A_1 \\ -A_2, & A_1, & 0 \end{pmatrix}$$

Entries A_a and $L_a^{\pm}$ are given as

$$L_a^{\pm} = -\frac{1}{2}\,\xi_a^R\,\coth(x_b - x_c) - \left(R_{am}\,\eta_m^L + \frac{1}{2}\,\xi_a^R\right)\tanh(x_b - x_c)$$
$$\pm\left(R_{am}\,\eta_m^L + \xi_a^R\right),$$

$$A_a = \frac{1}{2}\,\frac{\xi_a^R}{\sinh^2(x_b - x_c)} - \frac{R_{am}\,\eta_m^L + \frac{1}{2}\,\xi_a^R}{\cosh^2(x_b - x_c)}$$

Singular orbits. The case of $GL(3,\mathbb{R})$ and $GL(4,\mathbb{R})$

The motion on the Singular orbit is modified due to the continuous isotropy group. This symmetry of dynamical system is encoded in constraints on phase space variables

$$\psi_1 = \frac{1}{\sqrt{2}}\left(x_2 - x_3\right)\,, \qquad \psi_2 = \frac{1}{\sqrt{2}}\left(p_2 - p_3\right)\,,$$

$$\psi_3 = \xi_2^R - \xi_3^R, \qquad \psi_4 = \xi_1^R$$

One can check that this surface represents the invariant submanifold. These constraints are the second class in the Dirac terminology hence we have to replace the Poisson brackets by the Dirac ones

$$\{x_1, p_1\}_D = 1\,, \quad \{x_i, p_j\}_D = \frac{1}{2}\,, \qquad i,j = 2,3\,,$$
$$\{\xi^R_a, \xi^R_b\}_D = 0\,, \qquad a,b = 1,2,3$$

As a result we obtain

$$H^{(2)}_{GL(3,\mathbb{R})} = \frac{1}{2}p_1^2 + \frac{1}{4}p_2^2 + \frac{g^2}{\sinh^2(x_1 - x_2)}$$

This is an integrable mass deformation of the **IIA_1 Calogero-Moser-Sutherland model**.

The Lax pair for system can be obtained from the L and A matrices letting $\eta^R = 0$ and projecting on the constraint shell

$$L^{(2)}_{GL(3,\mathbb{R})} = \begin{pmatrix} \frac{1}{2}p_1, & -\xi_2^R L^+, & -\xi_2^R L^+ \\ \xi_2^R L^-, & \frac{1}{2}p_2 & 0 \\ \xi_2^R L^-, & 0, & \frac{1}{2}p_2 \end{pmatrix}$$

$$A^{(2)}_{GL(3,\mathbb{R})} = \frac{\xi_2^R}{\sinh^2(x_1 - x_2)} \begin{pmatrix} 0 & -1, & 1 \\ 1, & 0, & 0 \\ -1, & 0, & 0 \end{pmatrix}$$

Here

$$L^{-} := (1 - \coth(x_1 - x_2)) \ ,$$
$$L^{+} := (1 + \coth(x_1 - x_2))$$

Consider the case of $GL(4,\mathbb{R})$ group restricted on the Singular orbit with equal eigenvalues $x_3 = x_4$. The invariant submanifold is fixed by the five constraints

$$\begin{aligned}
\psi_1 &:= \frac{1}{\sqrt{2}}\left(x_3 - x_4\right) ,\\
\psi_2 &:= \frac{1}{\sqrt{2}}\left(p_3 - p_4\right) ,\\
\psi_3 &:= l^R_{34} ,\\
\psi_4 &:= l^R_{13} - l^R_{14} ,\\
\psi_5 &:= l^R_{23} - l^R_{24}
\end{aligned}$$

The Poisson matrix $\|\{\psi_m,\psi_n\}\|$, $m,n=1,\dots,5$ is degenerate with **rank**$\|\{\psi_m,\psi_n\}\|=4$. To find out the proper gauge and simplify the constraints it is useful to pass

$$l^R_{ab}=y_a\pi_b-y_b\pi_a\ ,\qquad \{y_a,\pi_b\}=\delta_{ab}\,,\qquad a,b=1,\dots,4$$

and choose the following gauge-fixing condition

$$\overline{\psi}:=\frac{1}{\sqrt{2}}\,(y_3-y_4)=0$$

Finally the reduced phase space corresponding to the Singular orbit is defined by the set of four second class constraints $\{\psi, \Pi, \overline{\psi}, \overline{\Pi}\}$

$$\psi := \frac{1}{\sqrt{2}}(x_3 - x_4) = 0\,, \qquad \Pi := \frac{1}{\sqrt{2}}(p_3 - p_4) = 0\,,$$

$$\overline{\psi} := \frac{1}{\sqrt{2}}(y_3 - y_4) = 0\,, \qquad \overline{\Pi} := \frac{1}{\sqrt{2}}(\pi_3 - \pi_4) = 0\,,$$

This set of constraints form an invariant submanifold of the phase space under the action of the discrete permutation group S_2

$$\begin{pmatrix} x_i \\ p_i \end{pmatrix} \mapsto \begin{pmatrix} x_{S(i)} \\ p_{S(i)} \end{pmatrix}, \qquad \begin{pmatrix} y_i \\ \pi_i \end{pmatrix} \mapsto \begin{pmatrix} y_{S(i)} \\ \pi_{S(i)} \end{pmatrix},$$

where $i = 3, 4$.

These constraints form the canonical set of second class constraints with non-vanishing Poisson brackets

$$\{\psi, \Pi\} = 1\,, \qquad \{\bar{\psi}, \overline{\Pi}\} = 1$$

and thus the fundamental Dirac brackets for canonical variables are

$$\{x_i, p_j\}_D = \frac{1}{2}\,, \qquad \{y_i, \pi_j\}_D = \frac{1}{2}\,, \qquad i,j = 1,2$$

The system reduces to the following one

$$H^{(3)}_{GL(4,\mathbb{R})} = \frac{1}{2}\,p_1^2 + \frac{1}{2}\,p_2^2 + \frac{1}{4}\,p_3^2 + \frac{(l^R_{12})^2}{16\sinh^2(x_1-x_2)} + \frac{(l^R_{13})^2}{8\sinh^2(x_1-x_3)} + \frac{(l^R_{23})^2}{8\sinh^2(x_2-x_3)}$$

with the Poisson bracket algebra

$$\{x_r\,, p_s\} = \delta_{rs}\,,$$
$$\{l^R_{pq}\,, l^R_{rs}\} = \delta_{ps}l^R_{qr} - \delta_{pr}l^R_{qs} + \delta_{qs}l^R_{pr} - \delta_{qr}l^R_{ps}\,, p, q, r, s = 1, 2, 3$$

In this case L and A matrices are the well-known matrices for the **spin Calogero-Sutherland** model

$$L_{ij} = \frac{1}{2}p_i\delta_{ij} - (1-\delta_{ij})l^R_{ij}\Phi(x_i - x_j)\,,$$
$$A_{ij} = -(1-\delta_{ij})l^R_{ij}V(x_i - x_j)\,,$$

where

$$\Phi(x) = \coth(x) + 1\,, \qquad V(x) = \frac{1}{2\sinh^2(x)}$$

After projection to the invariant submanifold corresponding to the motion on the **Singular orbits** we arrive at the Lax matrices

$$L^{(3)}_{GL(4,\mathbb{R})} = \begin{pmatrix} \frac{1}{2}p_1, & -l^R_{12}\Phi_{12}, & -l^R_{13}\Phi_{13}, & -l^R_{13}\Phi_{13} \\ -l^R_{12}\Phi_{12}, & \frac{1}{2}p_2, & -l^R_{23}\Phi_{23}, & -l^R_{23}\Phi_{23} \\ -l^R_{13}\Phi_{13}, & -l^R_{23}\Phi_{23}, & \frac{1}{2}p_3, & 0 \\ -l^R_{13}\Phi_{13}, & -l^R_{23}\Phi_{23}, & 0, & \frac{1}{2}p_3 \end{pmatrix}$$

$$A^{(3)}_{GL(4,\mathbb{R})} = \begin{pmatrix} 0, & l^R_{12}V_{12}, & l^R_{13}V_{13}, & l^R_{13}V_{13} \\ -l^R_{12}V_{12} & 0, & l^R_{23}V_{23}, & l^R_{23}V_{23} \\ -l^R_{13}V_{13}, & -l^R_{23}V_{23}, & 0, & 0 \\ -l^R_{13}V_{13}, & -l^R_{23}V_{23}, & 0, & 0 \end{pmatrix}$$

Geodesic flow on $gl(n, \mathbb{R})$

Consider the Lagrangian represented by the left-invariant metric

$$L = \frac{1}{2} < \dot{A}\,, \dot{A} >, \quad A \in GL(n, \mathbb{R})$$

which is defined on $gl(n, \mathbb{R})$ algebra

$$< X\,, Y >= \mathbf{Tr}\, X^T\, Y\,, \qquad X, Y \in gl(n, \mathbb{R})$$

The obviously conserved angular momentum

$$\mu = [A\,,\dot{A}]$$

leads to the possibility to integrate the equations of motion $\ddot{A} = 0$ in the form $A = at + b$. The canonical Hamiltonian corresponding to L is

$$H = \frac{1}{2}\,\mathbf{tr}\left(P^T P\right)^2$$

The nonvanishing Poisson brackets between the fundamental phase space variables are

$$\{A_{ab}\,, P_{cd}\} = \delta_{ab}\,\delta_{cd}$$

Using the same machinery as in the previous case we obtain the Hamiltonian

$$H = \frac{1}{2}\sum_{a=1}^{3} p_a^2 + \frac{1}{4}\sum_{(abc)} \frac{(\xi_a^R)^2}{(x_b - x_c)^2} + \sum_{(abc)} \frac{\left(R_{ab}\eta_b^L + \frac{1}{2}\xi_a^R\right)^2}{(x_b - x_c)^2}\,,$$

which is a generalization of the
rational Euler-Calogero-Moser-Sutherland model

After reduction on the invariant submanifold defined by the equations $\eta^R = 0$ we obtain the Hamiltonian

$$H = \frac{1}{2}\sum_{a=1}^{3} p_a^2 + \frac{1}{4}\sum_{(abc)} (\xi_a^R)^2 \left[\frac{1}{(x_b - x_c)^2} + \frac{1}{(x_b - x_c)^2}\right]$$

which coincides with the Hamiltonian of the **ID$_3$ Euler-Calogero-Moser-Sutherland model**

Forty years after the famous papers of

M. Toda

Journal of Physical Society of Japan 20 (1967) 431

Journal of Physical Society of Japan 20 (1967) 2095

The Hamiltonian of the non-periodic Toda lattice is

$$H_{NPT} = \frac{1}{2}\sum_{a=1}^{n} p_a^2 + g^2 \sum_{a=1}^{n-1} \exp\left[2(x_a - x_{a+1})\right],$$

and the Poisson bracket relations are

$$\{x_i, p_j\} = \delta_{ij}\,.$$

The equations of motion for the non-periodic Toda lattice are

$$\dot{x}_a = p_a,\ a = 1, 2, \ldots, n\,,$$
$$\dot{p}_a = -2\ \exp\left[2(x_a - x_{a+1})\right] + 2\ \exp\left[2(x_{a-1} - x_a)\right],$$
$$\dot{p}_1 = -2\ \exp\left[2(x_1 - x_2)\right],$$
$$\dot{p}_n = -2\ \exp\left[2(x_{n-1} - x_n)\right],$$

and for the periodic Toda lattice

$$\dot{x}_a = p_a\,,$$
$$\dot{p}_a = 2\ \exp\left[2(x_{a-1} - x_a)\right] - \exp\left[2(x_a - x_{a+1})\right].$$

Further for simplicity we put $k_{ab} = 0$ and $\alpha = 0$.

We use the Gauss decomposition for the positive-definite symmetric matrix S

$$S = Z\,D\,Z^T\,.$$

Here $D = \mathbf{diag}\,\|x_1, x_2, \ldots x_n\|$ is a diagonal matrix with positive elements and Z is an upper triangular matrix

$$Z = \begin{pmatrix} 1 & z_{12} & \ldots & z_{1n} \\ 0 & 1 & \ldots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

The differential of the symmetric matrix S is given by

$$dS = Z\left[dD + D\Omega + (D\Omega)^T\right] Z^T\,,$$

where Ω is a matrix-valued right-invariant 1-form defined by

$$\Omega := dZZ^{-1}\,.$$

In the Lie algebra $gl(n,\mathbb{R})$ of $n\times n$ real matrices we introduce Weyl basis with elements

$$(e_{ab})_{ij} = \delta_{ai}\,\delta_{bj}\,,$$

which are $n \times n$ matrices in the form

$$e_{ab} = \begin{pmatrix} & & \vdots & \\ \dots & \dots & \dots & 1 & \dots \\ & & \vdots & \\ & & \vdots & \\ & & \vdots & \end{pmatrix} .$$

The scalar product in $gl(n,\mathbb{R})$ is defined by

$$(e_{ab}\,, e_{cd}) = \mathbf{tr}(e^T_{ab}\,, e_{cd}) = \delta_{ac}\,\delta_{bd}\,.$$

Let us introduce also the matrices

$$\bar{\alpha}_a = \begin{pmatrix} 0 & \ldots & \ldots & \ldots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix}$$

and

$$\alpha_{ab} = \begin{pmatrix} & \vdots & & \vdots & \\ \ldots & \ldots & \ldots & 1 & \ldots \\ & \vdots & & \vdots & \\ \ldots & 1 & \ldots & \vdots & \ldots \\ & \vdots & & \vdots & \end{pmatrix}.$$

with scalar product is defined by

$$(\bar{\alpha}_a\,,\bar{\alpha}_b) = \mathbf{tr}(\bar{\alpha}_a\,\bar{\alpha}_b) = \delta_{ab}\,,$$
$$(\alpha_{ab}\,,\alpha_{cd}) = \mathbf{tr}(\alpha_{ab}\,\alpha_{cd}) = 2\,(\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd})\;,$$
$$(\bar{\alpha}_a\,,\alpha_{bc}) = \mathbf{tr}(\bar{\alpha}_a\,\alpha_{bc}) = 0\,.$$

Now we can write the differential of the matrix S in the form

$$dS = Z\left[\sum_{a=1}^{n} dx_a \bar{\alpha}_a + \sum_{a<b=1}^{n} x_a \Omega_{ab}\, \alpha_{ab}\right] Z^T\,.$$

Here Ω_{ab} are the coefficients of the matrix Ω

$$\Omega = \sum_{a<b} \Omega_{ab}\, e_{ab}\,.$$

In the case of Gauss decomposition of the symmetric matrix $S = ZDZ^T$ the corresponding momenta we seek in the form

$$P = (Z^T)^{-1}\left(\sum_{a=1}^{n}\bar{\mathcal{P}}_a\bar{\alpha}_a + \sum_{a<b}^{n}\mathcal{P}_{ab}\,\alpha_{ab}\right)Z^{-1}\,.$$

From the condition of invariance of the symplectic 1-form

$$\mathbf{tr}\,(PdS) = \sum_{i=1}^{n} p_a\, dx_a + \sum_{a<b=1}^{n} p_{ab}\, dz_{ab}\,,$$

where the new canonical variables $(x_a\,, p_a)\,, (z_{ab}\,, p_{ab})$ obey the Poisson bracket relations

$$\{x_a\,, p_b\} = \delta_{ab}\,, \qquad \{z_{ab}\,, p_{cd}\} = \delta_{ac}\,\delta_{bd}\,,$$

we obtain

$$\bar{\mathcal{P}}_a = p_a\,, \qquad \mathcal{P}_{ab} = \frac{l_{ab}}{2x_a}\,.$$

Here l_{ab} are right-invariant vector fields on the group of upper triangular matrices with unities on the diagonal

$$l_{ab} = \left(\Omega^{-1}\right)^T_{ab,cd} \, p_{cd} \, ,$$

where $\Omega_{ab,cd}$ are coefficients in the decompositions of the 1-forms Ω_{ab} in coordinate basis

$$\Omega_{ab} = \Omega_{ab,cd} \, dz_{cd} \, ,$$

The canonical Hamiltonian in the new variables takes the form

$$H = \frac{1}{2}\sum_{a=1}^{n} p_a^2\, x_a^2 + \frac{1}{4}\sum_{a<b=1}^{n} l_{ab}^2\, \frac{x_a}{x_b}\,.$$

After the canonical transformation

$$x_a = e^{y_a}\,, \qquad p_a = \pi_a\, e^{-y_a}$$

we arrive to Hamiltonian in the form of
spin nonperiodic Toda chain

$$H = \frac{1}{2}\sum_{a=1}^{n}\pi_a^2 + \frac{1}{4}\sum_{a<b=1}^{n} l_{ab}^2\, e^{y_a - y_b}\,.$$

The explicit form of the generators of the group of upper triangular matrices is

$$l_{12} = p_{12} + \sum_{k=2}^{n} z_{2k}\, p_{1k}\,, \qquad l_{13} = p_{13} + \sum_{k=2}^{n} z_{2k}\, p_{1k}\,.$$

The internal variables satisfy the Poison bracket relations

$$\{l_{ab}\,, l_{cd}\} = C^{ef}{}_{ab,cd}\, l_{ef} \tag{1}$$

with structure coefficients

$$C^{ef}{}_{ab,cd} = \delta_{bc}\,\delta_{ae}\,\delta_{df}\,. \tag{2}$$

1. Homogeneous cosmological models

1.1. *Spacetime decomposition*

Let $(\mathcal{M}, g)$ be a smooth four-dimensional paracompact and Hausdorf manifold. In each point of the open set $\mathcal{U} \subset \mathcal{M}$ we propose that are defined the local basis of 1-forms and its dual basis

$$\{\omega^\mu, \ \mu \ = 0, \ 1, \ 2, \ 3\} \qquad \{e_\nu, \ \nu \ = 0, \ 1, \ 2, \ 3\}$$

such that

$$[e_\alpha, e_\beta] = C^\mu{}_{\alpha\beta} e_\mu,$$

where $C^\mu{}_{\alpha\beta}$ are the basis structure functions. The symmetric metric is

$$g = g_{\mu\nu}\omega^\mu \otimes \omega^\nu \,, \qquad g^{-1} = g^{\mu\nu} e_\mu \otimes e_\nu$$

We suppose that on the manifold $(\mathcal{M}, g)$ is defined affine connection ∇

$$\Gamma^\mu{}_\nu = \Gamma^\mu{}_{\nu\alpha}\omega^\alpha \Longleftrightarrow \Gamma^\mu{}_{\beta\alpha} = \langle \omega^\mu, \nabla_{e_\alpha} e_\beta \rangle$$

In manifold with affine connection we can construct the

bilinear antisymmetric mapping

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X\,,Y]$$

The First Cartan structure equation is

$$d\omega^\mu + \Gamma^\mu{}_\alpha \wedge \omega^\alpha \; = \frac{1}{2} T^\mu$$

The tensor type $(1,3)$ defined by

$$R(\omega,\; Z,\; X,\; Y) = \langle \omega,\; R(\;X,\; Y)Z\rangle$$

is called the curvature tensor, where

$$R(\;X,\; Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - [\nabla_X, \nabla_Y]$$

If we define the curvature 2-form $\Omega^{\mu}{}_{\nu}$ by

$$\Omega^{\mu}{}_{\nu} = d\Gamma^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\alpha} \wedge \Gamma^{\alpha}{}_{\nu}$$

one can obtain the Second structure Cartan equation

$$\Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\alpha\beta}\, \omega^{\alpha} \wedge \omega^{\beta}$$

In the Riemannian geometry the commutator of two vector fields is given by

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

and for every vector field X we have the condition $\nabla_X g = 0$.

The connection and the Riemann tensor in the noncoordinate basis $\{e_\nu\}$ are given with

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} \left(e_\alpha g_{\beta\sigma} + e_\beta g_{\alpha\sigma} - e_\sigma g_{\alpha\beta} \right)$$
$$-\frac{1}{2} g^{\mu\sigma} (g_{\alpha\rho} C^\rho{}_{\beta\sigma} + g_{\beta\rho} C^\rho{}_{\alpha\sigma}) - \frac{1}{2} C^\mu{}_{\alpha\beta} ,$$
$$R^\sigma{}_{\alpha\mu\nu} = e_\mu \Gamma^\sigma{}_{\alpha\nu} - e_\nu \Gamma^\sigma{}_{\alpha\mu} + \Gamma^\rho{}_{\alpha\nu} \Gamma^\sigma{}_{\rho\mu} - \Gamma^\rho{}_{\alpha\mu} \Gamma^\sigma{}_{\rho\nu} - \Gamma^\sigma{}_{\alpha\rho} C^\rho{}_{\mu\nu} ,$$

where $C^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha} - \Gamma^\mu{}_{\alpha\beta}$.

Hereinafter we suppose that

$$\mathcal{M} = T_1 \times \Sigma_t$$

Let us introduce the noncoordinate basis of vector fields $\{e_\perp,\, e_a\}$

$$[e_\perp, e_a] = C^{\perp}{}_{\perp a} e_\perp + C^{d}{}_{\perp a} e_d \,,$$
$$[e_a, e_b] = C^{d}{}_{ab} e_d \,,$$

with the structure functions

$$C^{\perp}{}_{\perp a} = e_a ln N, \qquad C^{d}{}_{\perp a} = \frac{1}{N}(N^b C^{d}{}_{ab} + e_a N^d)$$

The metric in the corresponding dual basis $\{\theta^\perp,\, \theta^a\}$

$$g = -\theta^\perp \otimes \theta^\perp + \gamma_{ab}\, \theta^a \otimes \theta^b,$$

where γ is the induced metric on the submanifold Σ_t. The components of the vector field X_0 in this basis

$$X_0 = Ne_\perp + N^a e_a$$

are the Lagrange multipliers in ADM scheme which can be obtained if we pass to the coordinate basis $\{X_0 = \frac{\partial}{\partial t}, \frac{\partial}{\partial x^a}\}$

$$g = -\left(N^2 - N^a N_a\right) dt \otimes dt + 2N_a dt \otimes dx^a + \gamma_{ab} dx^a \otimes dx^b$$

The second fundamental form characterizes the embeding of

Σ_t in $(\mathcal{M}, g)$

$$K(X,Y) : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$$

$$K(X,Y) = \frac{1}{2}(Y.\nabla_X e_\perp - X.\nabla_Y e_\perp), \qquad X,Y \in T\Sigma_t$$

The another representation for the extrinsic curvature is

$$K_{ab} = -\frac{1}{2}\mathcal{L}_{e_\perp}\gamma_{ab}$$

To find the $3+1$ decomposition we define the induced on the Σ_t connection

$${}^3\Gamma^c{}_{ba} := \Gamma^c{}_{ba} = \langle {}^3\theta^c, {}^3\nabla_{e_a} e_b \rangle,$$

where ${}^3\nabla_X$ is the covariant derivative with respect to γ and corresponding curvature tensor is

$$ {}^3R(X,Y) = {}^3\nabla_X {}^3\nabla_Y - {}^3\nabla_Y {}^3\nabla_X - [{}^3\nabla_X , {}^3\nabla_Y] $$

In the basis $\{e_\perp , e_a\}$ we find the componets of the connection

$$
\begin{aligned}
\Gamma^{\perp}{}_{\perp\perp} &= 0, & \Gamma^{\perp}{}_{\perp i} &= 0, \\
\Gamma^{j}{}_{\perp\perp} &= c^j, & \Gamma^{j}{}_{\perp i} &= -K^j{}_i, \\
\Gamma^{\perp}{}_{i\perp} &= c_i, & \Gamma^{\perp}{}_{ji} &= -K_{ji}, \\
\Gamma^{j}{}_{i\perp} &= -K^j{}_i + \langle {}^3\theta^j, \mathcal{L}_{e_\perp} e_i \rangle, & \Gamma^{k}{}_{ij} &= {}^3\Gamma^k{}_{ij}
\end{aligned}
$$

and of the Riemann tensor

$$R^{j}{}_{\perp k \perp} = K^{js} K_{sk} + \gamma^{js} \mathcal{L}_{e_\perp} K_{ks} + \frac{1}{N} \gamma^{js} {}^3\nabla_{e_k} {}^3\nabla_{e_s} N \,,$$
$$R^{\perp}{}_{ijk} = {}^3\nabla_{e_k} K_{ij} - {}^3\nabla_{e_j} K_{ik} \,,$$
$$R^{s}{}_{ijk} = {}^3R^{s}{}_{ijk} + K_{ik} K_{j}{}^{s} - K_{ij} K_{k}{}^{s}$$

Finally the scalar curvature can be obtained in the form

$$R = {}^3R + K_a{}^a K_b{}^b - 3 K_{ab} K^{ab} - \frac{2}{N} \gamma^{ab} {}^3\nabla_{e_a} {}^3\nabla_{e_b} N - 2\gamma^{ab} \mathcal{L}_{e_\perp} K_{ab}$$

The classical behavior of the dynamical variables N, N^a, γ

is determined by the Hilbert-Einstein action

$$L[N, N_a, \gamma_{ab}, \dot{N}, \dot{N}_a, \dot{\gamma}_{ab}] = \iint\limits_{t_1 S}^{t_2} dt \ {}^3\sigma N \left\{ {}^3R + K_a{}^a K_b{}^b - 3K_{ab}K^{ab} \right\}$$

$$- \iint\limits_{t_1 S}^{t_2} dt \ {}^3\sigma N \left\{ \frac{2}{N} \gamma^{ab} \ {}^3\nabla_{e_a} \ {}^3\nabla_{e_b} N - 2\gamma^{ab} \ \mathcal{L}_{e_\perp} K_{ab} \right\}$$

where ${}^3\sigma = \sqrt{\gamma} \ \theta^1 \wedge \theta^2 \wedge \theta^3$ is the volume 3-form on Σ_t.

1.2. *Bianchi models*

L. Bianchi 1897

Sugli spazii atre dimensioni du ammettono un gruppo continuo di movimenti

Soc. Ital. della Sci. Mem. di Mat.

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By definition, Bianchi models are manifolds with product topology

$$\mathcal{M} = \mathbb{R} \times G_3$$

On the three dimensional Riemannnian manifold $\Sigma_t\gamma$ there

exist left-invariant 1-forms $\{\chi^a\}$ such that

$$d\chi^a = -\frac{1}{2} C^a{}_{bc}\, \chi^b \wedge \chi^c$$

The dual vector fields $\{\xi_a\}$ form a basis in the Lie algebra of the group G_3

$$[\xi_a, \xi_b] = C^d{}_{ab} \xi_d$$

with structure constants $C^d{}_{ab} = 2\, d\chi^d(e_a, e_b)$. In the invariant basis

$$[e_\perp, e_a] = C^d{}_{\perp a} e_d, \qquad [e_a, e_b] = -C^d{}_{ab} e_d,$$

with $C^d{}_{\perp a} = N^{-1} N^b C^d{}_{ab}$ the metric takes the form

$$g = -\,\theta^\perp \otimes \theta^\perp + \gamma_{ab}\, \theta^a \otimes \theta^b$$

The preferable role of this choice for a coframe is that the functions N, N^a and γ_{ab} depend on the time parameter t only. Due to this simplification the initial variational problem for Bianchi A models is restricted to a variational problem of the "mechanical" system

$$L\left(N, N_a, \gamma_{ab}, \dot{\gamma}_{ab}\right) = \int_{t_1}^{t_2} dt \sqrt{\gamma} N \left[{}^3R - K_a{}^a K_b{}^b + K_{ab} K^{ab} \right],$$

where ${}^3R$ is the curvature scalar formed from the spatial metric γ

$${}^3R = -\frac{1}{2}\gamma^{ab} C^c{}_{da} C^d{}_{cb} - \frac{1}{4}\gamma^{ab}\gamma^{cd}\gamma_{ij} C^i{}_{ac} C^j{}_{bd},$$

and

$$K_{ab} = -\frac{1}{2N}\Big((\gamma_{ad} C^d{}_{bc} + \gamma_{bd} C^d{}_{ac}) N^c + \dot{\gamma}_{ab} \Big)$$

is the extrinsic curvature of the slice Σ_t defined by the relation

$$K_{ab} = -\frac{1}{2}\mathcal{L}_{e_\perp}\gamma_{ab}$$

In the theory we have four primary constraints

$$\pi := \frac{\delta L}{\delta \dot{N}} = 0\,, \qquad \pi^a := \frac{\delta L}{\delta \dot{N}_a} = 0$$

$$\pi^{ab} := \frac{\delta L}{\delta \dot{\gamma}_{ab}} = \sqrt{\gamma}(\gamma_{ab} K^i{}_i - K^{ab})$$

The symplectic structure on the phase space is defined by the following non-vanishing Poisson brachets

$$\{N, \pi\} = 1, \qquad \{N_a, \pi^b\} = \delta_a^b, \qquad \{\gamma_{cd}, \pi^{rs}\} = \frac{1}{2}\Big(\delta_c^r \delta_d^s + \delta_c^s \delta_d^r\Big)$$

Due to the reparametrization symmetry of inherited from the diffeomorphism invariance of the initial Hilbert-Einstein action, the evolution of the system is unambiguous and it is governed by the total Hamiltonian

$$H_T = N\mathcal{H} + N^a \mathcal{H}_a + u_0 P^0 + u_a P^a$$

with four arbitrary functions $u_a(t)$ and $u_0(t)$. One can verify that the secondary constraints are first class and obey the algebra

$$\{\mathcal{H}, \mathcal{H}_b\} = 0, \quad \{\mathcal{H}_a, \mathcal{H}_b\} = -C^d{}_{ab}\mathcal{H}_d$$

From the condition of time conservation of the primary constaraints folllows four secondary constaraints

$$\mathcal{H} = \frac{1}{\sqrt{\gamma}}\left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^a{}_a\pi^b{}_b\right) - \sqrt{\gamma}\ {}^3R, \qquad \mathcal{H}_a = 2\ C^d{}_{ab}\pi^{bc}\gamma_{cd},$$

which obey the algebra

$$\{\mathcal{H}, \mathcal{H}_a\} = 0, \qquad \{\mathcal{H}_a, \mathcal{H}_b\} = -C^d{}_{ab}\mathcal{H}_d$$

The Hamiltonian form of the action for the Bianchi A models can be obtained in the form

$$L[N, N_a, \gamma_{ab}, \pi, \pi^a, \pi^{ab}] = \int_{t_1}^{t_2} \pi^{ab} d\gamma_{ab} - H_C dt$$

where the canonical Hamiltonian is a linear combination of the secondary constraints $H_C = N\mathcal{H} + N^a\mathcal{H}_a$.

1.3. *Hamiltonian reduction of Bianchi I cosmology*

In the case of Bianchi I model the group which acts on (Σ_t, γ) is $\mathbf{T}^3$ and the action takes the form

$$L[N, N_a, \gamma_{ab}, \pi, \pi^a, \pi^{ab}] = \int_{t_1}^{t_2} \pi^{ab} d\gamma_{ab} - N\mathcal{H}dt\,.$$

Using the decomposition for arbitrary symmetric non-singular matrix

$$\gamma = R^T(\chi) e^{2X} R(\chi)\,,$$

where $X = diag\|x_1, x_2, x_3\|$ is diagonal matrix and

$$R(\psi, \theta, \phi) = e^{\psi J_3} e^{\theta J_1} e^{\phi J_3}$$

we can pass to the new canonical variables

$$(\gamma_{ab}, \pi^{ab}) \Longrightarrow (\chi_a, p_{chi_a}; x_a, p_a)$$

The corresponding momenta are

$$\pi = R^T \left(\sum_{s=1}^{3} \overline{\mathcal{P}}_s \, \overline{\alpha}_s + \sum_{s=1}^{3} \mathcal{P}_s \, \alpha_s \right) R$$

where

$$\overline{\mathcal{P}}_a = p_a \ ,$$
$$\mathcal{P}_a = -\frac{1}{4}\frac{\xi_a}{\sinh(x_b - x_c)}, \quad (cyclic \ \ permutations \ \ a \neq b \neq c)$$

and the left-invariant basis of the action of the $SO(3,\mathbb{R})$ in the phase space with three dimensional orbits is given by

$$\xi_1 = \frac{\sin\psi}{\sin\theta}\ p_\phi + \cos\psi\ p_\theta - \sin\psi\cot\theta\ p_\psi,$$
$$\xi_2 = -\frac{\cos\psi}{\sin\theta}\ p_\phi + \sin\psi\ p_\theta + \cos\psi\cot\theta\ p_\psi\ ,$$
$$\xi_3 = p_\psi$$

In the new variables the Hamiltonian constraint reads

$$\mathcal{H} = \frac{1}{2}\sum_{a=1}^{3} p_a^2 - \sum_{a<b} p_a p_b + \frac{1}{2}\sum_{(abc)} \frac{\xi_c^2}{\sinh^2(x_a - x_b)}$$

Many thanks

TO THE ORGANIZERS

for the nice Conference !!!

Also many thanks to all of you !!!

I hope we will see each other
in the next edition of the Conference . . .