NONCOMMUTATIVE CHERN-SIMONS SOLITONS

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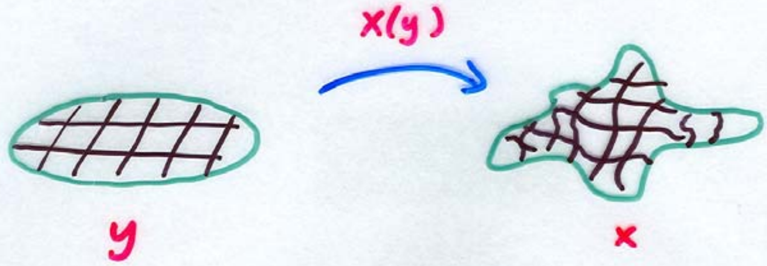
ELECTRON FLUID



$$x^i = y^i + \varepsilon^{ik} \frac{A_k}{2\pi\rho_0}$$

$$L = \int \tfrac{1}{2} m \dot{x}^2 \longrightarrow L = \int F_{ik} F^{ik}$$

Area preserv. diffeomorph. $\longleftrightarrow$ Gauge transf.

Density fluct. $\longleftrightarrow$ Photon

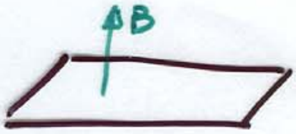
Velocity $\longleftrightarrow$ $\bar{E}$

$\partial \cdot x$ $\longleftrightarrow$ $\bar{B}$

No fluid vort. $\longleftrightarrow$ $\bar{\nabla} \cdot \bar{E} = 0$ (conserved)

No fluid displacement $\longleftrightarrow$ $\bar{\nabla} \times \bar{A} = 0$

add magnetic field



additional term

$$L' = \frac{eB}{2}\rho_0 \int \varepsilon^{ik} \dot{x}_i x_k$$

or

$$L' = \frac{eB}{8\pi^2 \rho_0} \int \varepsilon^{ik} \dot{A}_i A_k \quad \text{(lowest order)}$$

Abelian Chern-Simons

+ constraint

$$\frac{1}{2} \varepsilon^{ij} \varepsilon^{ab} \partial_i x_a \partial_j x_b = 1$$

or

$$\nabla \times \bar{A} = 0 \quad \text{(fluid displacement)}$$

together gives (first terms of)

Noncommutative Chern-Simons

$$L = \varepsilon^{\alpha\beta\gamma} \int d^2y \left(A_\alpha \partial_\beta A_\gamma - \frac{2i}{3} A_\alpha * A_\beta * A_\gamma \right)$$

$\theta = \frac{1}{2\pi\rho_0}$ "star product"

Noncommutative Chern-Simons

$$[\hat{x}, \hat{y}] = -i\theta$$

$$\Rightarrow \partial_x = \frac{1}{i\theta}[\hat{y}, \cdot\,] \quad \partial_y = -\frac{1}{i\theta}[\hat{x}, \cdot\,]$$

$$\partial_i = \varepsilon_{ik} \frac{1}{i\theta}[\hat{x}^k, \cdot\,]$$

"covariant" position

cov. der.

$$\hat{\underline{X}}^i = \hat{x}^i - \theta \varepsilon^{ik} \hat{A}_k = i\theta \varepsilon^{ik} D_i$$

under gauge transformations

$$\hat{\underline{X}}^i \to U^+ \hat{\underline{X}}^i U$$

in this notation

$$L_{CS} = -\frac{\pi\kappa}{\theta} \mathrm{Tr}\left(-\varepsilon_{ij} \hat{\underline{X}}^i (\dot{\hat{\underline{X}}}^j - i[A_0, \hat{\underline{X}}^j]) + 2\theta A_0\right)$$

complex notation

$$\hat{c} = \frac{1}{\sqrt{2\theta}}(\hat{x}^1 - i\hat{x}^2) \qquad [\hat{c}, \hat{c}^+] = 1$$

$$\hat{K} = \frac{1}{\sqrt{2\theta}}(\hat{\underline{X}}^1 - i\hat{\underline{X}}^2)$$

$$L_{cs} = i\pi\kappa \, Tr(K^\dagger D_t K - K D_t K^\dagger) - 2\pi\kappa \, Tr A_0$$

e.o.m.'s

δA_0: $\quad [K, K^\dagger] = 1$

δK: $\quad D_t K = 0$

solution

$$K = c$$

Corresponds to (SW-map) Seiberg, Witten Okawa Ooguri

$$\tilde{\rho}(k) \equiv Tr(e^{-ik\mathbb{X}})$$

$$\rho(x) = \int \frac{d^2k}{(2\pi)^2} \tilde{\rho}(k) e^{ikx}$$

$K = c$ (or $\mathbb{X} = x$) implies

$$\tilde{\rho}(k) = \frac{2\pi}{\theta} \delta^{(2)}(k)$$

$$\rho(x) = \frac{1}{2\pi\theta} \quad \text{(constant density)}$$

Comment on SW-map

Okawa
Ooguri

$\mathrm{tr}\left(e^{-ik\cdot\hat{x}}\right)$ not gauge inv.

$\Rightarrow \mathrm{tr}\left(e^{-ik\cdot\hat{X}}\right)$

Remember M(atrix) theory

Eigenvals of X ~ pos. of constituents (D0-branes)

$$\tilde{\rho}(k) = \mathrm{tr}\left(e^{-ikX}\right) = \sum_m e^{-ikx_m} = \int d^2x\, \rho(x) e^{-ik\cdot x}$$

for $\rho(x) = \sum_m \delta(x - x_m)$

Using this philosophy

Center:

$$R^i = \int d^2x \left[\int \frac{d^2k}{(2\pi)^2} \operatorname{Tr}\left(\hat{X}^i e^{-ik\cdot\hat{X}}\right) e^{ikx} \right]$$

Size:

$$\Delta^2 = \int d^2x \left[\int \frac{d^2k}{(2\pi)^2} \operatorname{Tr}\left((\hat{X}^i - R^i)^2 e^{-ik\hat{X}} \right) e^{ikx} \right]$$

We need more interesting solutions!

$$[K, K^\dagger] = 1 + \underbrace{q|0\rangle\langle 0|}$$

$$D_t K = 0$$

"source" term added by hand

Solution

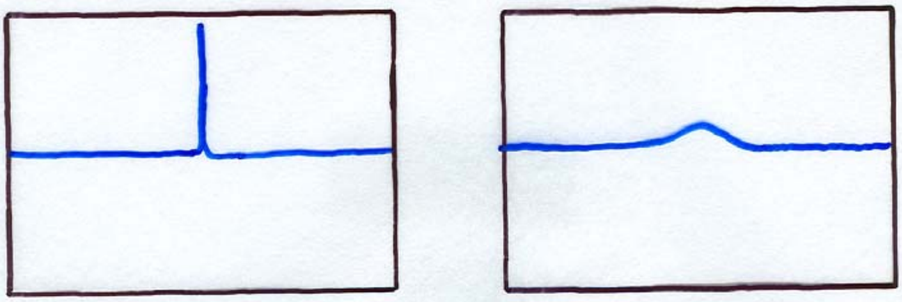
$$K|n\rangle = \sqrt{n+q}\,|n-1\rangle$$

$$K|0\rangle = 0$$

$$K^\dagger|n\rangle = \sqrt{n+1+q}\,|n+1\rangle$$

(8)

density (numerically)



$q \ll 1$ $\qquad$ $q = 1$

NO DYNAMICS!

Try again: (Bak)

$$L = L_{CS} + 2\pi\theta \, Tr\left\{ iD_t\phi\,\phi^\dagger - \frac{1}{2\theta}\left(D\phi\,\bar{D}\phi^\dagger + \bar{D}\phi\,D\phi^\dagger\right) + \frac{1}{2\kappa}(\phi\phi^\dagger)^2 \right\}$$

$$\left\{\begin{array}{l} D_t\phi = \dot{\phi} - iA_0\phi \\ D\phi = K\phi - \phi c \\ \bar{D}\phi = -(K^\dagger\phi - \phi c^\dagger) \end{array}\right.$$

BPS eqn.

$$[K, K^\dagger] = 1 - \underbrace{\frac{\theta}{\kappa}\phi\phi^\dagger}_{\text{source term}}$$

$$K^\dagger\phi - \phi c^\dagger = 0$$

$$A_0 = -\frac{1}{2\kappa}\phi\phi^\dagger$$

solutions:

(Bak)

$$\phi = \sqrt{\frac{\kappa}{\theta}}\, |0\rangle\langle z|$$

coherent state

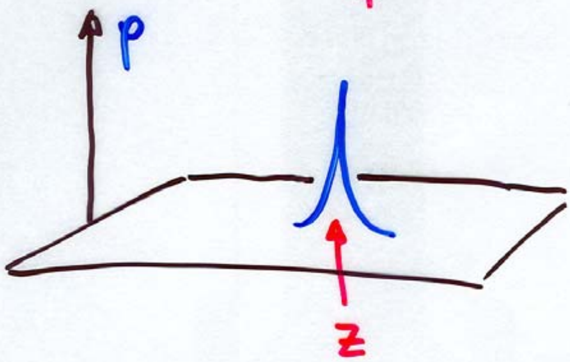
$$K = z|0\rangle\langle 0| + \hat{S}_1 \hat{C} \hat{S}_1^{+} \qquad \hat{S}_1 = \sum |n+1\rangle\langle n|$$

$$\Rightarrow \quad \frac{\theta}{\kappa}\phi\phi^{+} = |0\rangle\langle 0|$$

so $\quad [K, K^{+}] = 1 - |0\rangle\langle 0|$

$q = -1$

now



multivortex:

(Bak)

$$\phi = \sqrt{\frac{m\kappa}{\theta}}\, |m-1\rangle\langle 0|$$

$$K = \hat{P}_m \hat{C} \hat{P}_m + \hat{S}_m \hat{C} \hat{S}_m^{+}$$

$$[K, K^{+}] = 1 - m\,|m-1\rangle\langle m-1|$$

charge m source

$\Delta^2 = \theta(m-1)$ size minimal

symmetries:

Gauge:

$$\begin{cases} \phi \to U\phi \\ K \to UKU^{\dagger} \\ A_0 \to U(i\partial_t + A_0)U^{\dagger} \end{cases}$$

U unitary $UU^{\dagger} = 1$

Translation:

$$\begin{cases} \phi \to \phi T^{\dagger}(z) \\ K \to K + z \leftarrow \text{shifts density in SW-map} \\ A_0 \to A_0 \end{cases}$$

$T = e^{z\hat{c}^{\dagger} - \bar{z}\hat{c}}$ unitary

can calculate Noether current

ILL DEFINED $(\sim Tr(K))$

happens also in commutative thy. (Manton)

Solution: Trans $\Rightarrow$ Trans + Gauge

Translation: (gauge covariant form)

$$\begin{cases} \phi \to \tilde{T}(z)\,\phi\, T(z)^{\dagger} \\ K \to \tilde{T}(z)\, K\, \tilde{T}(z)^{\dagger} + z \\ A_0 \to \tilde{T}(z)(i\partial_t + A_0)\tilde{T}(z)^{\dagger} \end{cases}$$

transl. with $T(z) = e^{z\hat{C}^{+} - \bar{z}\hat{C}}$

+ gauge transf. with $\tilde{T}(z) = e^{z\hat{K}^{+} - \bar{z}\hat{K}}$

the conserved current (momentum) is

$$P = \pi\theta \,\mathrm{Tr}(\hat{J})$$

$\hat{J} = \frac{i}{2}\left(\phi(\bar{D}\phi)^{+} - (D\phi)\phi^{+}\right)$ charge current

mom. density ~ charge current

Rotations: (already gauge covariant)

$$\begin{cases} \phi \Rightarrow \tilde{R}\,\phi\, R^{+} \\ K \to e^{-i\alpha}\,\tilde{R}\, K\, \tilde{R}^{+} \\ A_0 \to \tilde{R}(i\partial_t + A_0)\tilde{R}^{+} \end{cases}$$

$$R = e^{-\frac{i}{2}\alpha\{c, c^{+}\}} \qquad \tilde{R} = e^{-\frac{i}{2}\alpha\{K, K^{+}\}}$$

Conserved charge (angular momentum)

$$Q = 2\pi\theta\, \mathrm{Tr}\Big(i(K^{+}J - KJ) - \tfrac{1}{2}\big(|D\phi|^2 + |\bar{D}\phi|^2\big)\Big)$$

Galilean boosts: (gauge covariant)

(Horvathy, Martina, Stichel)

$$\phi \to T_1\, \phi\, T_2^{+}$$

$$K \to T_1(K + v t)\, T_1^{+}$$

$$A_0 \to T_1\big(A_0 + i\big(v(K^{+} - c^{+}) - \bar{v}(K - c)\big)\big)T_1^{+}$$

$$T_1 = e^{(v\hat{c}^{+} - \bar{v}\hat{c})t} \qquad T_2 = e^{(v(t - i\theta)\hat{c}^{+} - \bar{v}(t + i\theta)\hat{c})}$$

v (complex) velocity

θ noncommutativity parameter

Energy:

$$E = \pi \mathrm{Tr}\left(D\phi (D\phi)^\dagger + \bar{D}\phi(\bar{D}\phi)^\dagger - \frac{\theta}{k}(\phi\phi^\dagger)^2 \right)$$

under a boost, E transforms

$$E \to E + 2\pi \mathrm{Tr}\left(\theta(\bar{v}J + vJ^\dagger) + |v|^2\theta^2\phi\phi^\dagger \right)$$

What happens to sol: solutions under boosts?

Ex:

$$\phi = \sqrt{\tfrac{k}{\theta}}|0\rangle\langle 0| \to \sqrt{\tfrac{k}{\theta}}|0\rangle\langle \underbrace{v(t-i\theta)}_{\text{coherent state}}|$$

$$K = \hat{S}_1 \hat{c}\, \hat{S}_1^\dagger \to \hat{S}_1 \hat{c}\, \hat{S}_1 + vt$$

$$A_0 = -\frac{1}{2\theta}|0\rangle\langle 0| \to -\frac{1}{2\theta}|0\rangle\langle 0| + i\hat{S}_1(v\hat{c}^\dagger - \bar{v}\hat{c})\hat{S}_1^\dagger$$

Before boost

$$P = 0 \quad Q = -\pi K \quad E = 0$$

After boost

$$P = \pi k\theta v \quad Q = -\pi k(1 + \underbrace{2\theta^2|v|^2}_{?}) \quad E = \underbrace{2\pi k\theta|v|^2}_{\text{Kinetic energy!}}$$

Not through origin

$$Q = -\pi k\left(1 + 2\theta^2|v|^2 + \underbrace{i\theta(\bar{v}z_0 - v\bar{z}_0)}_{\bar{r}\times\bar{p}}\right)$$

2-soliton dynamics:

$$\phi = \sqrt{\frac{2\kappa}{\theta}}\left(A|0\rangle\langle +| + B|1\rangle\langle -|\right)$$

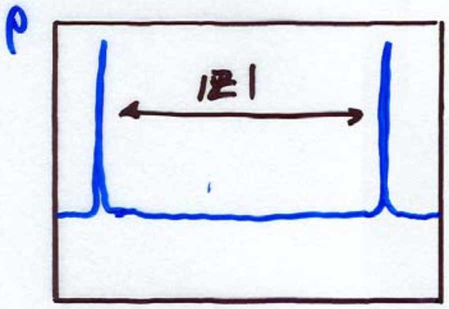
$$K = z\left(C|0\rangle\langle 1| + \frac{1}{C}|1\rangle\langle 0|\right) + S_2 \, c \, S_2^{+}$$

z rel. dist. of solitons

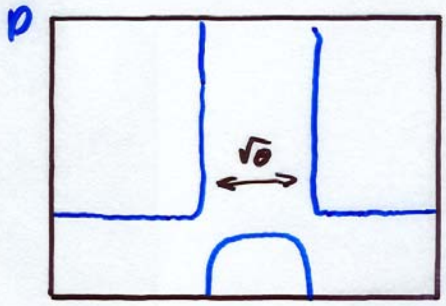
$$|\pm\rangle = N_{\pm}\left(|z\rangle \pm |-z\rangle\right)$$

$$\begin{cases} |z|^2\left(C^2 - \frac{1}{C^2}\right) = B^2 - A^2 \\ A^2 + B^2 = 1 \end{cases}$$

Profile:



$|z| \gg \sqrt{\theta}$



$|z| \ll \sqrt{\theta}$

Manton's method:

$$z \to z(t) \to S(z(t))$$

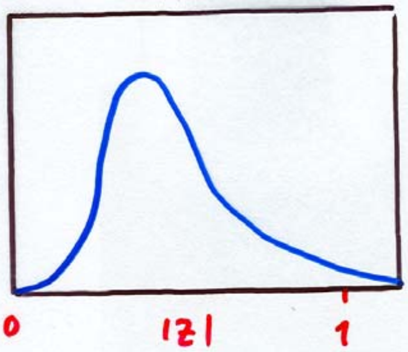
effective action

$$i\pi\kappa \int dt \left(c^2 - \frac{1}{c^2}\right)^2 |z|^2 (\bar{z}\dot{z} - z\dot{\bar{z}})$$

first order ⇒ no nontrivial dynamics

⇒ symplectic form

$$\begin{cases} \omega = d\mathcal{A} \\ \mathcal{A} = i\left(c^2 - \frac{1}{c^2}\right)^2 |z|^2 (z\,d\bar{z} - \bar{z}\,dz) \end{cases}$$



commutative ω is Kähler form for moduli space metric of Maxwell-Higgs vortex solutions, here not!

Other stuff

new solutions (non-BPS)

$$\begin{cases} \phi = \sqrt{\frac{\kappa}{\theta}}\,|0\rangle\langle n|\, e^{-in\frac{t}{\theta}} \\ \kappa = S_1 c S_1^{+} \\ A_0 = -\frac{1}{2\theta}\,|0\rangle\langle 0| \end{cases}$$

Exactly same density profile as original solitons but

$$Q = -\pi\kappa(1+2n)\,, \quad E = \frac{2\pi\kappa}{\theta}\, n$$

Rotationally invariant!

Carry some kind of internal spin.

Conclusions

- NCCS + nr. scalar introduce dynamic vortices
- ∃ hole spectrum of interesting soliton solutions
- Dynamics ?
- Quantization ?
- Properties of multisolitons ?
- Relation to Maxwell-Higgs ?
- Exact SW-map for fractionally charged solitons ?
- Relation to FQHE ?