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A non-perturbative approach

to

non-commutative scalar field theory

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Motivation and relevance

1. Intuitive:

   Standard model of high-energy physics based on spacetime-continuum idealization, seems unplausible;

   $\Rightarrow$ quantized space? Schrödinger, Heisenberg

   (cp. string theory, quantum gravity)

   possible realization: space is noncommutative (NC)

2. Analytical: Physics in a NC world: new analytical tools:

   matrix model techniques?

   spinoffs: strong magnetic fields, QHE, incompressible fluids

3. Phenomenological:

   physical consequences of a quantization of space?

   new types of models, new features (to be explored):

   preferred gauge group $U(n)$, symmetry breaking, UV/IR relation, etc

How to proceed?

Non-commutative geometry, field theory

- Manifold $\mathcal{M} \rightarrow$ NC algebra $\mathcal{A}$ of functions on $\mathcal{M}$

  simplest example $\mathbb{R}^n_\theta$:

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij}$$

  θ_{ij} ... a.s. tensor (background field)

  (cp. Quantum Mechanics, phase space)

  usually $\exists$ derivatives ∂_i, integral, some symmetries

- Field theory on NC space:

$$\begin{aligned} \mathcal{C}(\mathcal{M}) &\rightarrow \mathcal{A} \\ f(x) &\rightarrow \hat{f}(\hat{x}) \end{aligned}$$

  (e.g. plane waves $e^{ikx} \rightarrow : e^{ik\hat{x}} :$).

  Formulation of field theory is possible, many examples

NC scalar field theory

consider some NC space, algebra $\mathcal{A}$ (e.g. $\mathbb{R}^d_\theta, T^2_\theta, S^2_N, \mathbb{C}P^2_N, ...$)

use representation of algebra $\mathcal{A}$ on Hilbert space $\mathcal{H}$

Field $\phi(x) \rightsquigarrow \phi \in L(\mathcal{H})$ ... Hermitian operator on Hilbert space $\mathcal{H}$

trace replaces integral

<u>Example:</u>

$$S = \sqrt{det(2\pi\theta)} Tr \left(\frac{1}{2}\partial_i\phi\partial_i\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{4}\phi^4 \right)$$

can write e.g. $\phi(x) = \int dk\, \phi_k : e^{ikx} :$ etc.,

Quantization

formally defined by (Euclidean) path integral

$$\langle \phi_{k_1} \cdots \phi_{k_l} \rangle = \frac{\int [\mathcal{D}\Phi] e^{-S}\, \phi_{k_1} \cdots \phi_{k_l}}{\int [\mathcal{D}\Phi] e^{-S}}, \qquad [\mathcal{D}\Phi] = \prod d\phi_k$$

⇒Wick's theorem, however distinction planar ↔ nonplanar diagrams

propagator: as usual, $\langle \phi_k \phi_{k'} \rangle = \delta_{kk'} \frac{1}{k^2+m^2}$

one-loop planar and non-planar self-energy diagrams:

$$\begin{aligned} \Gamma_P^{(2)} &:= g \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m^2} \sim g\Lambda^{d-2}, \\ \Gamma_{NP}^{(2)}(p) &:= g \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\theta p}}{k^2+m^2} \sim g \left(\frac{1}{1/\Lambda^2 + p^2\theta^2} \right)^{\frac{d-2}{2}} \end{aligned}$$

$\Gamma_{NP}^{(2)}(p)$ is finite as long as $p \neq 0$, but IR singularity as $p \to 0$

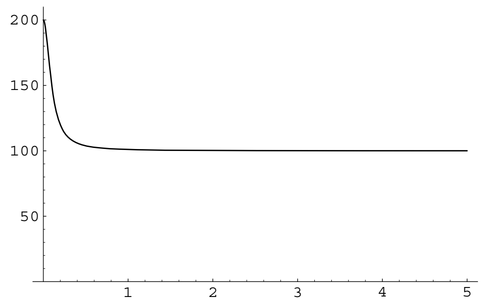
... UV/IR mixing *(Minwalla, Van Raamsdonk, Seiberg)*

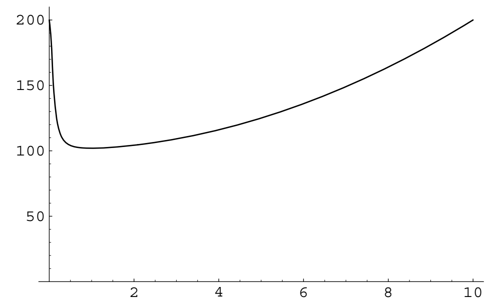
central feature of NC field theories,

serious obstacle to perturbative renormalization

Note:

Λ fixed, large: $\Gamma^{(2)}_{NP}(p)$ $\Gamma^{(2)}(p)$





$\Rightarrow$low-momentum modes are "stiff", suppressed (UV/IR)

expect SSB to $\langle p \rangle \neq 0$ ... "striped" phase *(Gubser, Sondhi)*

verified numerically *(Ambjorn, Catterall; Bietenholz, Hofheinz, Nishimura; Martin)*

"matrix" phase *(Martin)* numerical

Matrix regularization of NCFT

consider compact NC spaces, e.g. fuzzy $\mathbb{C}P^n_N$, NC tori, ...

then $dim(\mathcal{H}) \sim$ Vol = finite,

locally $\boxed{\mathbb{C}P^n_N \rightsquigarrow \mathbb{R}^{2n}_\theta}$ if scale $\frac{R^2}{N} \sim \theta_{ij}$ $\quad (as\ \ R, N \to \infty)$

<u>action:</u>

$$S = \int_{\mathbb{C}P^n} \left(\frac{1}{2}\phi\Delta\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4}g\phi^4 \right)$$

were $\phi \in Mat(\mathcal{N}, \mathbb{C})$ ... Hermitian matrix ($\mathcal{N} \approx N^n$)

<u>note</u>:

- same propagator as for $\mathbb{R}^{2n}_\theta$, cutoff $|p| \leq \Lambda \approx \sqrt{N/\theta}$
- matrix model, but kinetic term breaks $U(\mathcal{N})$ invariance $\phi \to U^{-1}\phi U$
- quantization: $Z = \int D\phi\, e^{-S}$ formally well-defined

change of variables: $\phi = U^{-1} diag(\phi_i) U, \qquad U \in U(\mathcal{N})$

$$\int D\phi = \int d\phi_i \Delta^2(\phi_i) \int dU$$

where $\Delta^2(\phi_i) = \prod_{i<j}(\phi_i - \phi_j)^2$ ... Jacobian

idea: study the eigenvalue sector of this model: integrate out dU,

$$\begin{aligned} Z &= \int D\phi \; \exp(-S(\phi))) \\ &= \int d\phi_i \Delta^2(\phi_i) \int dU \exp(-S(U^{-1}(\phi_i)U)) \\ &= \int D\phi \exp(-\tilde{\mathcal{F}}(\phi) - (2\pi\theta)^{d/2}\, TrV(\phi)) \end{aligned}$$

where

$$\tilde{F}(\phi) := \int dU \exp(-S_{kin}(U^{-1}(\phi)U)) =: e^{-\tilde{\mathcal{F}}(\phi)}$$

depends only on eigenvalues, analytic.

$\Rightarrow$ effective matrix model for the eigenvalue sector:

$$\tilde{S}(\phi) = \tilde{\mathcal{F}}(\phi) + (2\pi\theta)^{d/2}\, TrV(\phi_i)$$

Claim: in NC regime $\Lambda^2\theta \gg 1$

the following holds:

- the path integral induces a measure in the space of eigenvalues ϕ_i which is sharply localized due to UV/IR mixing,

  eigenvalue sector can be analyzed using the saddle-point method

- for "weak coupling", the eigenvalue sector is described by the *effective matrix model*

$$\tilde{S}(\phi) = f_0(m) + \frac{2\mathcal{N}}{\alpha_0^2(m)} Tr\phi^2 + S_{int}(\phi)$$

where

$$\alpha_0^2(m) = 4c\Lambda^{d-2} = \begin{cases} \frac{1}{4\pi^2}\,\Lambda^2\left(1 - \frac{m^2}{\Lambda^2}\ln(1+\frac{\Lambda^2}{m^2})\right), & d=4 \\ \frac{1}{\pi}\ln(1+\frac{\Lambda^2}{m^2}), & d=2 \end{cases}$$

is the scale of the eigenvalues, which are distributed in the interval

$$\phi_i \in [-\alpha_0, \alpha_0].$$

The free case

Claim:

$$\tilde{S}_0(\phi) = f_0(m) + \frac{2\mathcal{N}}{\alpha_0^2(m)} Tr\phi^2$$

Strategy of Proof: consider the observables $\frac{1}{V}\int d^dx\phi^{2n}(x) = \frac{1}{\mathcal{N}}Tr\phi^{2n}$ and their products, compute expectation values, show that $\tilde{S}_0(\phi)$ reproduces same result.

Start:

$$\langle\frac{1}{V}\int d^dx\phi^2(x)\rangle = \int_0^\Lambda \frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+m^2} =: \alpha_0^2(m)$$

Next, consider

$$\langle\tfrac{1}{V}\int d^dx\phi(x)^4\rangle \quad = \langle\tfrac{1}{V}\int d^dx\phi(x)^4\rangle_P + \langle\tfrac{1}{V}\int d^dx\phi(x)^4\rangle_{NP}$$

there are 2 planar and one non-planar contractions;

The non-planar contribution is sub-leading:

introduce rescaled field $\phi = \alpha_0 \varphi$, $\langle \frac{1}{\mathcal{N}} Tr \varphi^2 \rangle = \frac{1}{4}$. then

$$\langle \frac{1}{V} \int d^d x \varphi(x)^{2n} \rangle_{NP} \approx \int_0^1 \frac{d^d k'_1}{(k'_1)^2} ... \frac{d^d k'_n}{(k'_n)^2} e^{i\Lambda^2 \sum k'_i \theta k'_j} \to 0 \qquad \text{as } \Lambda \to \infty,$$

vanishes for large Λ, due to the rapidly oscillating exponential. Therefore

$$\langle \frac{1}{V} \int d^d x \varphi^{2n} \rangle = \langle \frac{1}{V} \int d^d x \varphi^{2n} \rangle_{Planar} = N_{Planar}(2n) \qquad \text{as } \Lambda \to \infty$$

replace $\frac{1}{V} \int d^d x = \frac{1}{\mathcal{N}} Tr$, use factorization: $\Rightarrow$

$$\boxed{\langle (\frac{1}{\mathcal{N}} Tr \varphi^{n_1}) ... (\frac{1}{\mathcal{N}} Tr \varphi^{n_k}) \rangle = N_{Planar}(2n_1) \; ... \; N_{Planar}(2n_k)}$$

known: Gaussian matrix model:

$$S_{eff} = 2\mathcal{N} Tr \varphi^2$$

where φ ... $\mathcal{N} \times \mathcal{N}$ matrix reproduces the same result (for $\mathcal{N} \to \infty$)

$\Rightarrow$effective action for eigenvalue sector of free NC field theory (obtained by integrating out dU) is given by Gaussian Matrix Model

$$\boxed{\tilde{S}_0 = f_0(m) + \frac{2\mathcal{N}}{\alpha_0^2} Tr\phi^2}$$

Describes correctly the eigenvalue sector of the field theory.

$f_0(m)$ ... numerical function of m, can be chosen such that

$$Z = \int D\phi \exp(-f_0(m) - \frac{2\mathcal{N}}{\alpha_0^2} Tr\phi^2)$$

Known: EV distribution of $\tilde{S}_0$ is dominated by saddle-point

$$(\varphi_1 \leq ... \leq \varphi_{\mathcal{N}}) \quad \rightarrow \quad \varphi(s) = \varphi_j, \qquad s = \frac{j}{\mathcal{N}}$$

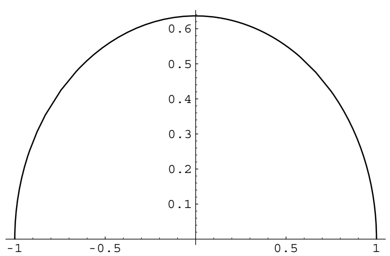
... EV distribution function in continuum limit $\mathcal{N} \rightarrow \infty$

useful: $\varphi(s) \leftrightarrow$ eigenvalue density ρ:

$$ds = \rho(\varphi)d\varphi, \qquad \int_{-\infty}^{\infty} \rho(p)dp = 1.$$

Solution: Wigner's semi-circle law

$$\rho(p) = \begin{cases} \frac{2}{\pi}\sqrt{1-p^2} & p^2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$



i.e.: eigenvalues of ϕ are distributed in $[-\alpha_0, \alpha_0]$

path integral is dominated by eigenvalue distribution $\rho(p)$.

Interactions

add

$$S_{int}(\phi) = \frac{g_n}{n} \int d^d x \phi^n(x)$$

above results generalize at least on a perturbative level:

$$Z_{int} = \int D\phi e^{-(S_0+S_{int})} = \int D\phi e^{-S_0} \left(1 - \frac{g_n}{n}(2\pi\theta)^{d/2} Tr\phi^n + ...\right)$$

Therefore: the eigenvalue sector of the NCFT with action

$$S = \int d^d x \frac{1}{2}(\partial_i\phi\partial_i\phi + m^2\phi^2) + S_{int}$$

is described by the *effective matrix model*

$$\boxed{\tilde{S}(\phi) = f_0(m) + \frac{2\mathcal{N}}{\alpha_0^2(m)} Tr\phi^2 + S_{int}(\phi)}$$

established if EV distribution is close to Wigner's law: *weak coupling*.

extends naturally to non-perturbative domain: matrix model techniques!

Interaction $\Rightarrow$ modification of eigenvalue distribution semi-classical picture:

classical fluid $\phi(x)$ on a compact space (e.g. $\mathbb{C}P^n$)

fixed EV distribution $\leftrightarrow$ field $\phi(x)$ has prescribed "density"

$$\rho(p) = \frac{1}{V} \int d^d x \, \delta(\phi(x) - p),$$

corresponding to the eigenvalue distribution.

$\Rightarrow$expect unusual long-distance effects (UV/IR)

Example: the ϕ^4 model

Consider

$$
\begin{aligned}
S &= \int d^dx \left(\tfrac{1}{2}\partial_i\phi\partial_i\phi + \tfrac{1}{2}m^2\phi^2 + \tfrac{g}{4}\phi^4\right) \\
&= \int d^dx \left(\tfrac{1}{2}\partial_i\phi\partial_i\phi + \tfrac{1}{2}m_R^2\phi^2 + \left(\tfrac{1}{2}\Delta m^2\phi^2 + \tfrac{g}{4}\phi^4\right)\right)
\end{aligned}
$$

$m_R^2 = m^2 - \Delta m^2$ ... "renormalized" mass

Eigenvalue sector according to the above results described by

$$
\begin{aligned}
\tilde{S} &= f_0(m_R) + \tfrac{2\mathcal{N}}{\alpha_0(m_R)^2}\, Tr\phi^2 + (2\pi\theta)^{d/2} Tr\left(\tfrac{1}{2}\Delta m^2\phi^2 + \tfrac{g}{4}\phi^4\right) \\
&= f_0(m_R) + \mathcal{N} Tr\left(\tfrac{m'^2}{2}\varphi^2 + \tfrac{g'}{4} Tr\varphi^4\right)
\end{aligned}
$$

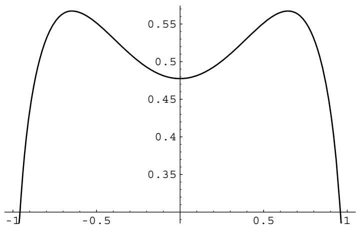
rescale field:

$$
\phi = \alpha_g\varphi, \quad \varphi \in [-1, 1]
$$

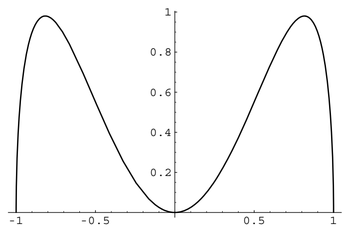
solve with standard methods from random matrix theory

2 phases:

1. "single-cut", weak coupling: $\rho_g(\varphi) = \frac{1}{2\pi}(g'\varphi^2 + \frac{g'}{2} + m'^2)\sqrt{1^2 - \varphi^2}$.



2. "2-cuts", strong coupling: eigenvalue density $\rightarrow$ 2 disjoint peaks, concentrated around the 2 minima.



semi-classical interpretation: 2-fluid model, large surface energy at contact surface due to the kinetic term.

$\Rightarrow$ will develop some kind of domains ("stripes"?): "striped" phase (Gubser & Sondhi), or equivalently "matrix phase" (Martin)

weak coupling and renormalization

"weak coupling": keep the eigenvalue distribution close to Wigner's law.

(close to free model!).

Renormalization:

how to scale the bare parameters m^2, g with Λ resp. $\mathcal{N}$ such that "low-energy physics" remains invariant

- turn on coupling $g \;\; \Rightarrow \;\;$ EV distribution (size α_g!) modified
- $\Rightarrow$must adjust m^2 (strongly negative!) to stay close to free EV distribution with

$$\alpha_g = \alpha_0(m_R)$$

  for finite m_R (EV distri determines correlation length, mass m_R)

this determines "renormalized mass" m_R (correlation length)

4 dimensions:

quadratic divergence of the bare mass

$$m^2 = m_R^2 - \frac{3}{16\pi^2}\,\Lambda^2\left(1 - \frac{m_R^2}{\Lambda^2}\,\ln(\frac{\Lambda^2}{m_R^2})\right)\,g.$$

agrees perfectly with a conventional one-loop computation from 8 planar plus 4 non-planar contractions.

coupling:

$$g' = \frac{2}{\pi^2}\left(1 - 2\,\frac{m_R^2}{\Lambda^2}\ln(\frac{\Lambda}{m_R})\right)^2\,g \approx \frac{2}{\pi^2}\,g$$

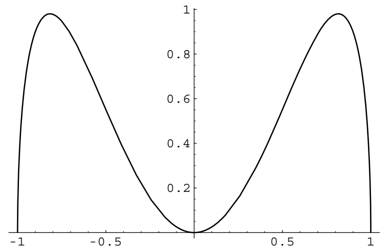
expected to be reliable (no expansion!) as long as $g' \ll 1$.

Note:

- is systematic, non-perturbative result from matrix model
  however: matching $\alpha_g = \alpha_0(m_R)$ is an approximation
- suggests existence of scaling limit with “near-free” properties

The phase transition

happens at $m'^4 = 4g'$.



conjecture: is phase transition between disordered and striped phase

2-fluid-interpretation, repulsion at surface $\Rightarrow$ pattern

4 dimensions:

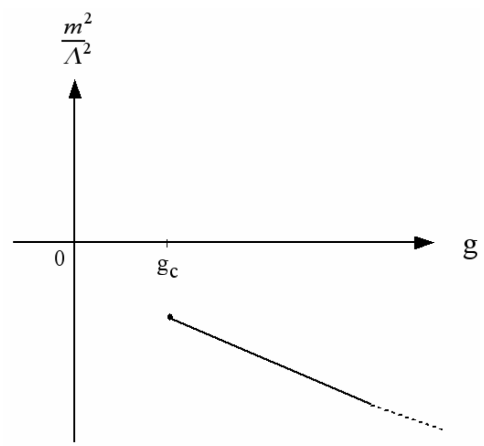
critical line:

$$2\pi\sqrt{\frac{2}{g}} = 1 - \left(\frac{m^2}{\Lambda^2} + \frac{3}{4\pi}\sqrt{\frac{g}{2}}\right)\ln\left(1 + \left(\frac{m^2}{\Lambda^2} + \frac{3}{4\pi}\sqrt{\frac{g}{2}}\right)^{-1}\right).$$

rhs is $\in (0,1] \Rightarrow$ has solution only for

$$g \geq g_c = 8\pi^2, \quad m_c^2/\Lambda^2 = -\frac{3}{2}$$

Non-trivial end-point of critical line



Note:

- non-trivial critical point expected generically
  (EV-distri. on critical line distinct from free case)
- higher-order phase transition (Gubser etal argued 1st order)
- expect critical point $\to$ fixed-point under RG flow
  should be a *non-trivial* model (different EV distri!)
- treat interactions exactly, approximate kinetic term
  (opposite of Gubser & Sondhi, Chen & Wu, Castorina & Zappala ...)

numerical checks: 2 dimensions (fuzzy sphere S_N^2)

Basic assumptions more delicate (only logarithmic divergence; require finite $\frac{m_R}{m_\theta}$)

above analysis *not quite applicable* ($m_R \to 0$; not allowed in 2D)

... go ahead anyway, find

$$\frac{m^2}{2N} = -\frac{3}{2\sqrt{\pi}}\,\frac{1}{R} \approx -0.846\,\frac{1}{R}.$$

for large $R \sim \sqrt{N}$ ... radius of S_N^2

numerical results (Martin): disordered-matrix transition at

$$\frac{m^2}{2N} \approx -0.56\,\frac{1}{R}$$

for large R.

Conclusion and outlook

- NC scalar field theory has sharp EV distribution due to UV/IR mixing,

  useful characterization of NCFT
- very relevant to interactions, renormalization,

  can apply sophisticated matrix model techniques
- analytical description of phase transition to “matrix / striped” phase:

  non-trivial critical point / fixed point in ϕ^4 model in 4D
- generalization to Higgs sector (complex scalar fields)?

  similar description of phase transition expected

  suggests new physics