A Gravity Theory on Noncommutative Spaces

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Outline

- Deformed Spaces and Symmetries
- Bialgebra of Deformed Diffeomorphisms
- Representations: Scalar, Vector and Tensor Fields
- Metric and Christoffel-Symbol
- Covariant Derivatives and Curvature
- θ-Deformed Einstein-Hilbert action
- Outlook

Deformed Spaces

- **Underlying idea:** Discrete Spacetime
  At very short distances: coordinates do not commute
  (Heisenberg 1930)

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}(\hat{x}) \neq 0$$

- differentiable space-time manifold $\longrightarrow$ algebra of noncommutative coordinates:

$$\widehat{\mathcal{A}}_{\theta(x)} := \mathbb{C}\langle\langle \hat{x}^1, \ldots, \hat{x}^n \rangle\rangle / ([\hat{x}^\mu, \hat{x}^\nu] - \theta^{\mu\nu}(\hat{x}))$$

- We consider elements of $\widehat{\mathcal{A}}_{\theta(x)}$ as “functions” on the deformed Space
- Easiest example: θ**-Deformed** or **Canonically Deformed Quantum Space** $\widehat{\mathcal{A}}_\theta$

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad \theta^{\mu\nu} \in \mathbb{R}$$

Star Product Realization

- Vector spaces of formal power series in commutative coordinates and noncommutative coordinates are isomorphic *(Poincaré-Birkhoff-Witt property)*

$$\begin{aligned} \rho : \mathbb{C}[x^0, \ldots, x^n][[\theta]] &\rightarrow \widehat{\mathcal{A}}_\theta \\ f(x^\mu) &\mapsto \hat{f}(\hat{x}^\mu) \end{aligned}$$

- We define a *new product*, called **star product** by pulling back the product of the noncommutative algebra:

$$f(x^\mu) \star g(x^\mu) := \rho^{-1}(\hat{f}(\hat{x}^\mu) \cdot \hat{g}(\hat{x}^\mu))$$

- This way we realized the deformed algebra $\widehat{\mathcal{A}}_\theta$ on the algebra of commutative functions equipped with a new *(noncommutative)* product $\star$

Star Product Realization

- The star product is *not* unique since the isomorphism ρ *(ordering prescription)* is not unique
- *Example:* Canonical Structure $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$
  Moyal-Weyl Product

$$f \star g = \mu \circ e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}(f \otimes g) = fg + \frac{i}{2}\theta^{\mu\nu}(\partial_\mu f)(\partial_\nu g) + \ldots,$$

  where $\mu(f \otimes g) := fg$ is just the multiplication map and

$$[x^\mu \stackrel{\star}{,} x^\nu] = i\theta^{\mu\nu}$$

- For $\theta \to 0$ we get back the commutative product
- The star product above is associative

Derivatives

- **Derivatives** are maps on the deformed coordinate space:

$$\hat{\partial}_\mu : \widehat{\mathcal{A}}_\theta \to \widehat{\mathcal{A}}_\theta$$

- Thus, they have to be consistent with the commutation relations of the coordinates.
- general ansatz:

$$[\hat{\partial}_\mu, \hat{x}^\nu] = \delta^\nu_\mu + \sum_j A_\mu^{\nu\rho_1\ldots\rho_j} \hat{\partial}_{\rho_1} \ldots \hat{\partial}_{\rho_j} \,.$$

- <u>θ-Deformed Space:</u> The *undeformed* calculus

$$[\hat{\partial}_\mu, \hat{x}^\nu] := \delta^\nu_\mu$$

is consistent with the algebra relations $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$

$\star$-Action and θ-Deformed Diffeomorphisms

- **Diffeomorphisms** are generated by vector fields $\xi = \xi^\mu \partial_\mu$ which satisfy the Lie algebra

$$[\xi, \eta] = \xi \times \eta = \big(\xi^\mu (\partial_\mu \eta^\nu) - \eta^\mu (\partial_\mu \eta^\nu)\big)\partial_\nu$$

- Aim: Lift vector fields to differential operators which define maps on the algebra $\widehat{\mathcal{A}}_\theta$, i.e. are consistent with the commutation relations of the coordinates
- Since we have a well-defined differential calculus also the following map

$$\hat{\xi} : \hat{f} \mapsto \hat{\xi}^\rho \hat{\partial}_\rho \hat{f}$$

  is a well-defined map on the deformed space $\widehat{\mathcal{A}}_\theta$

- Mapped to the functions of commuting variables we obtain the $\star$**-action**:

$$\xi : f \mapsto \xi^\rho \star (\partial_\rho f) =: \xi \triangleright f$$

$\star$-Action and θ-Deformed Diffeomorphisms

But: The $\star$-commutator of two such operators does not close and we don't have a representation of the Lie algebra of vector fields:

$$[\xi \stackrel{\star}{,} \eta] = \xi \times \eta = \big(\xi^\mu \star (\partial_\mu \eta^\nu) - \eta^\mu \star (\partial_\mu \eta^\nu)\big)\partial_\nu + [\xi^\mu \stackrel{\star}{,} \eta^\nu]\partial_\mu \partial_\nu$$

- Define higher order differential operator by requiring

$$[X_\xi \stackrel{\star}{,} X_\eta] = X_{[\xi,\eta]}$$

  where $\xi = \xi^\rho \partial_\rho$ is an ordinary vector field. Then the operators X_ξ have the Lie algebra structure of vector fields!

- Solution: $X_\xi = \xi - \frac{i}{2}\theta^{\mu\nu}(\partial_\mu \xi)\partial_\nu + \dots$
- Mapped to the space of commutative coordinates we get $X_\xi \triangleright f = \xi f = \xi^\mu (\partial_\mu f)$

$\star$-*Action and* θ-*Deformed Diffeomorphisms*

- The operators X_ξ act in a nontrivial way on the $\star$-product of two functions:

$$X_\xi \triangleright (f \star g) \;=\; (X_\xi \triangleright f) \star g + f \star (X_\xi \triangleright g) \\ -\frac{i}{2}\theta^{\rho\sigma}\{((\partial_\rho X_\xi) \triangleright f) \star (\partial_\sigma g) + (\partial_\rho f) \star ((\partial_\sigma X_\xi) \triangleright g)\} + O(\theta^2)$$

- One can abstract this deformed Leibniz rule to a nontrivial **coproduct**

$$\Delta X_\xi = X_\xi \otimes 1 + 1 \otimes X_\xi - \frac{i}{2}\theta^{\rho\sigma}\{((\partial_\rho X_\xi) \otimes \partial_\sigma + \partial_\rho \otimes (\partial_\sigma X_\xi)\} + O(\theta^2)$$

- The coproduct tells us how to act on a product of two functions

Scalar, Vector and Tensor Fields

- We introduce scalar fields to be elements ϕ in $\widehat{\mathcal{A}}_\theta$ with the following transformation property:

$$\delta_\xi \phi = -X_\xi \triangleright \phi$$

- The coproduct

$$\Delta\delta_\xi = \delta_\xi \otimes 1 + 1 \otimes \delta_\xi - \frac{i}{2}\theta^{\rho\sigma}\{((\partial_\rho\delta_\xi)\otimes\partial_\sigma + \partial_\rho\otimes(\partial_\sigma\delta_\xi)\} + O(\theta^2)$$

assures that a $\star$-product of scalar fields transforms as a scalar again:

$$\delta_\xi(\phi \star \psi) = -X_\xi \triangleright (\phi \star \psi)$$

- Similarly we define vector fields to transform like

$$\delta_\xi V_\mu = -X_\xi \triangleright V_\mu - X_{\xi^\rho_\mu} \star V_\rho$$

and tensor fields in full analogy

Summary: θ-Deformed Diffeomorphisms

- To summerize, we end up with the
  θ-Deformed Diffeomorphisms:
  The algebra is undeformed

$$[\delta_\xi, \delta_\eta] = \delta_{\xi\times\eta}$$

  but the coalgebra is deformed

$$\Delta(\hat{\delta}_\xi) = e^{-\frac{i}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma}\left(\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi\right)e^{\frac{i}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma}$$

- We have scalar fields and tensor fields as representations of it
- On scalar fields the above deformed algebra is represented by the differential operators X_ξ from the previous discussion

Metric and Christoffel-Symbel

- We define the metric to be a symmetric tensor of rank two which reduces to the classical metric $g_{\mu\nu}$ for $\theta \to 0$
- This can be done using a set of four vectors ${E_\mu}^a$ **(Vierbeins)**

$$G_{\mu\nu} = \frac{1}{2}\Big({E_\mu}^a \star {E_\nu}^b + {E_\nu}^a \star {E_\mu}^b\Big)\eta_{ab},$$

where η_{ab} is the usual metric of flat Minkowski space

- The deformed coproduct assures that $G_{\mu\nu}$ transforms indeed as a tensor
- To raise and lower indices we need its $\star$-inverse $G^{\mu\nu\star}$

$$G_{\mu\nu} \star G^{\nu\rho\star} = \delta_\mu^\rho$$

Metric and Christoffel-Symbol

- This is again a tensor in $\mathcal{A}_\theta$ and can be constructed in terms of the classical inverse $G^{\mu\nu}$
- Let us assume that the connection is symmetric $\Gamma_{\mu\nu}{}^{\alpha} = \Gamma_{\nu\mu}{}^{\alpha}$
- Demanding that the covariant derivative of $G_{\mu\nu}$ vanishes

$$D_\alpha G_{\beta\gamma} = \partial^\star_\alpha \triangleright G_{\beta\gamma} - \Gamma^\rho_{\alpha\beta} \star G_{\rho\gamma} - \Gamma^\rho_{\alpha\gamma} \star G_{\beta\rho} = 0$$

we obtain $\Gamma_{\mu\nu}{}^{\alpha}$ uniquely expressed in terms of $G_{\mu\nu}$ and $G^{\mu\nu\star}$ **(Christoffel-Symbol)**

$$\Gamma^\sigma_{\alpha\beta} = \frac{1}{2}\Big(\partial^\star_\alpha \triangleright G_{\beta\gamma} + \partial^\star_\beta \triangleright G_{\alpha\gamma} - \partial^\star_\gamma \triangleright G_{\alpha\beta}\Big) \star G^{\gamma\sigma\star}$$

Covariant Derivatives and Curvature

- We are able to construct covariant derivatives

$$\mathcal{D}_\mu V_\nu := \partial_\mu V_\nu - \Gamma_{\mu\nu}{}^\alpha \star V_\alpha$$

such that $\mathcal{D}_\mu V_\nu$ transforms as a tensor

- The curvature tensor can be obtained as the commutator of two covariant derivatives

$$[\mathcal{D}_\mu \overset{\star}{,} \mathcal{D}_\nu] V_\rho = R_{\mu\nu\rho}{}^\sigma \star V_\sigma + T_{\mu\nu}{}^\alpha \star \mathcal{D}_\alpha V_\rho$$

- The curvature can be expressed in terms of the connection and can be expanded in orders of θ:

$$\begin{aligned} R_{\mu\nu\rho}{}^\sigma \quad = \quad & R^{(0)}_{\mu\nu\rho}{}^\sigma - \frac{i}{2}\theta^{\kappa\lambda}(\partial_\kappa R^{(0)}_{\mu\nu\rho}{}^\tau)(\partial_\lambda g_{\tau\gamma})g^{\gamma\delta} \\ & + \frac{i}{2}\theta^{\kappa\lambda}(\partial_\kappa \Gamma^{(0)}_{\nu\rho}{}^\beta)(\Gamma^{(0)}_{\mu\beta}{}^\tau(\partial_\lambda g_{\tau\gamma})g^{\gamma\sigma} - \Gamma^{(0)}_{\mu\tau}{}^\sigma(\partial_\lambda g_{\beta\gamma})g^{\gamma\tau} + (\partial_\lambda \Gamma^{(0)}_{\mu\beta}{}^\sigma)) \\ & - \frac{i}{2}\theta^{\kappa\lambda}(\partial_\kappa \Gamma^{(0)}_{\mu\rho}{}^\beta)(\Gamma^{(0)}_{\nu\beta}{}^\tau(\partial_\lambda g_{\tau\gamma})g^{\gamma\sigma} - \Gamma^{(0)}_{\nu\tau}{}^\sigma(\partial_\lambda g_{\beta\gamma})g^{\gamma\tau} + (\partial_\lambda \Gamma^{(0)}_{\nu\beta}{}^\sigma)) + \ldots , \end{aligned}$$

where $R^{(0)}_{\mu\nu\rho}{}^\sigma$ is the commutative curvature tensor

Deformed Einstein-Hilbert Action

- We can define the $\boldsymbol{\theta}$**-deformed Einstein-Hilbert action**

$$S_{\mathrm{EH}} = \frac{1}{2}\int \mathrm{d}^4x \left(E^\star \star R + \mathrm{c.c.}\right)$$

- It is real by definition and invariant with respect to the deformed diffeomorphisms
- Since the integral is cyclic

$$\int \mathrm{d}^4x \; f \star g = \int \mathrm{d}^4x \; g \star f$$

we can write (if the measure $E^\star$ is real):

$$S_{\mathrm{EH}} = \frac{1}{2}\int \mathrm{d}^4x \, (E^\star \star R + \bar{R} \star E^\star) = \frac{1}{2}\int \mathrm{d}^4x \; E^\star \star (R + \bar{R})$$

Discussion and Outlook

- We have constructed a θ-Deformed Bialgebra of Diffeomorphisms.
- We have studied scalar, vector and tensor fields as representations of this θ-Deformed Bialgebra of Diffeomorphisms.
- We introduced a covariant derivative and the curvature tensor
- We defined a metric and got the curvature scalar
- We construct a θ-deformed diffeomorphism invariant Einstein-Hilbert action and got

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. . . see hep-th/0504183 for more details