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Lorentz symmetry on deformed spaces

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Deformed (noncommutative) space

Commutative space generated by coordinates x^μ, $\mu = 1, \dots n$ with

$$[x^\mu, x^\nu] = 0.$$

Noncommutative space $\hat{\mathcal{A}}_{\hat{x}}$, generated by $\hat{x}^\mu$ coordinates, $\mu = 1, \dots n$ such that:

$$[\hat{x}^\mu, \hat{x}^\nu] = \Theta^{\mu\nu}(\hat{x}). \tag{1}$$

It is the associative free algebra generated by $\hat{x}^\mu$ and divided by ideal generated by relations (1).

In other words, it consists of all polynomials of $\hat{x}^\mu$ and if one polynomial can be transformed to another one using (1) they are considered equal.

$$\begin{aligned} \hat{x}^1\hat{x}^2 &= [\hat{x}^1, \hat{x}^2] + \hat{x}^2\hat{x}^1 \\ &= \Theta^{12}(\hat{x}) + \hat{x}^2\hat{x}^1 \end{aligned}$$

Three forms of $\Theta^{\mu\nu}(\hat{x})$ of special interest

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad \theta^{\mu\nu} = -\theta^{\nu\mu} = const., \tag{2}$$

$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\lambda \hat{x}^\lambda, \quad C^{\mu\nu}_\lambda \text{ Lie algebra struct. const.,} \tag{3}$$

$$\hat{x}^\mu \hat{x}^\nu = \frac{1}{q} R^{\mu\nu}{}_{\rho\sigma} \hat{x}^\rho \hat{x}^\sigma \quad R \text{ matrix of quantum group.} \tag{4}$$

θ-deformed space

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{5}$$
$$\theta^{\mu\nu} = -\theta^{\nu\mu} = const. \text{ and } \theta^{\mu\nu} \in \mathbb{R}.$$

Derivatives

Map $\hat{\partial}$ such that $$\hat{\partial}: \quad \hat{\mathcal{A}}_{\hat{x}} \to \hat{\mathcal{A}}_{\hat{x}} \tag{6}$$

defined on coordinates

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta^\mu_\rho + \text{ additional terms} \tag{7}$$

it has to be consistent with (5) relations

$$\hat{\partial}_\rho\Big([\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu}\Big) = 0. \tag{8}$$

Solution

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta^\mu_\rho. \tag{9}$$

Also,

$$[\hat{\partial}_\rho, \hat{\partial}_\sigma] = 0 \tag{10}$$

is compatible with (9),

$$\hat{\partial}_\sigma\Big([\hat{\partial}_\rho, \hat{x}^\mu] - \delta^\mu_\rho\Big) = \Big([\hat{\partial}_\rho, \hat{x}^\mu] - \delta^\mu_\rho\Big)\hat{\partial}_\sigma = 0.$$

Leibniz rule

$$\hat{\partial}_\rho\big(\hat{f}\hat{g}\big) = (\hat{\partial}_\rho\hat{f})\hat{g} + \hat{f}(\hat{\partial}_\rho\hat{g}). \tag{11}$$

Hopf algebra of derivatives

algebraic relations

$$[\hat{\partial}_\rho, \hat{\partial}_\sigma] = 0, \tag{12}$$

comultiplication

$$\begin{aligned}\Delta\hat{\partial}_\rho &= \hat{\partial}_\rho \otimes 1 + 1 \otimes \hat{\partial}_\rho, \\ (\Delta \otimes \mathsf{id}) \circ \Delta &= (\mathsf{id} \otimes \Delta) \circ \Delta \ \text{ and } \ [\Delta\hat{\partial}_\mu, \Delta\hat{\partial}_\nu] = 0,\end{aligned} \tag{13}$$

counit and antipode

$$\varepsilon(\hat{\partial}_\rho) = 0, \qquad S(\hat{\partial}_\rho) = -\hat{\partial}_\rho. \tag{14}$$

$\star$-product approach

Idea: express everything in terms of commutative variables, but keep track of deformation

Using Poincaré-Birkhoff-Witt (PBW) theorem

$$\hat{f}(\hat{x}) \mapsto f(x) \tag{15}$$

and

$$\hat{f}(\hat{x})\hat{g}(\hat{x}) = \hat{f} \cdot \hat{g}(\hat{x}) \mapsto f \star g(x), \tag{16}$$

where

$$f \star g\,(x) = e^{\frac{i}{2}\frac{\partial}{\partial x^\rho}\theta^{\rho\sigma}\frac{\partial}{\partial y^\sigma}} f(x)g(y)\Big|_{y\to x}, \tag{17}$$

or

$$f \star g\,(x) \;=\; \sum_{n=1}^{\infty}\left(\frac{i}{2}\right)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\dots\partial_{\rho_n} f(x)\Big)\Big(\partial_{\sigma_1}\dots\partial_{\sigma_n} g(x)\Big) \tag{18}$$

is Moyal-Weyl $\star$-product
-associative but noncommutative
-under complex conjugation

$$\overline{f \star g} = \bar{g} \star \bar{f}. \tag{19}$$

Special example $\quad [\hat{x}^\mu, \hat{x}^\nu] \mapsto [x^\mu \stackrel{\star}{,} x^\nu] = i\theta^{\mu\nu}$.

$\star$-product representation of derivatives

$$\begin{aligned} \hat{f} \cdot \hat{g}(\hat{x}) &\mapsto f \star g(x) \\ \hat{\partial}_\mu \hat{f}(\hat{x}) &\mapsto \underbrace{\partial^\star_\mu}_{?} f(x) \end{aligned}$$

How to do it:

$$\begin{array}{ccc} \hat{f}(\hat{x}) & \longmapsto & f(x) \\ \hat{\partial}\downarrow & & \downarrow \partial^\star \\ (\hat{\partial}\hat{f})(\hat{x}) & \longmapsto & (\partial^\star f)(x) \end{array} \tag{20}$$

For derivatives defined by (12)

$$(\partial^\star_\rho f)(x) = \partial_\rho f(x) \tag{21}$$

and

$$\partial^\star_\rho \Big(f \star g\,(x) \Big) = \Big(\partial^\star_\rho f(x) \Big) \star g(x) + f(x) \star \Big(\partial^\star_\rho g(x) \Big). \tag{22}$$

With (18), (21) and (22) one has enough ingredients to be able to start doing some physics.

Deformed Lorentz transformations

classical Lorentz transformations $x^\mu \to x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$, $\omega^{\mu\nu} = -\omega^{\nu\mu} = const.$

scalar field $\phi(x) \to \phi'(x') = \phi(x)$,

$$\delta^{cl}_\omega \phi(x) \stackrel{\text{def}}{=} \phi'(x) - \phi(x) = -\omega^\lambda{}_\nu x^\nu \partial_\lambda \phi(x). \tag{23}$$

Algebra of transformations

$$[\delta^{cl}_\omega, \delta^{cl\prime}_\omega] = \delta^{cl}_{[\omega,\omega']}, \tag{24}$$

Leibniz rule

$$\begin{aligned} \delta^{cl}_\omega \Big(\phi_1(x)\phi_2(x)\Big) &= \Big(\delta^{cl}_\omega \phi_1(x)\Big)\phi_2(x) + \phi_1(x)\Big(\delta^{cl}_\omega \phi_2(x)\Big) \\ &= -\omega^\lambda{}_\nu x^\nu \partial_\lambda \Big(\phi_1(x)\phi_2(x)\Big). \end{aligned} \tag{25}$$

Idea: deform transformations (23) in such a way that can be easily rewritten in terms of elements form $\hat{\mathcal{A}}_{\hat{x}}$, that is, lifted to the space of noncommutative coordinates.

Rewrite $\omega^\lambda{}_\nu x^\nu \partial_\lambda \phi(x)$ in terms of $\star$-product and $\star$-derivatives

$$\omega^\lambda{}_\nu x^\nu \partial_\lambda \phi = (X^\star_\omega \star \phi). \tag{26}$$

Solving (26) perturbatively one finds to all orders in θ

$$X^\star_\omega = (\omega^\lambda{}_\nu x^\nu)\partial^\star_\lambda - \frac{i}{2}\theta^{\rho\sigma}\omega^\lambda{}_\rho \partial^\star_\sigma \partial^\star_\lambda. \tag{27}$$

Deformed Lorentz transformation of a scalar field is

$$\begin{aligned} \delta_\omega \phi &= -X^\star_\omega \star \phi \\ &= -(\omega^\lambda{}_\nu x^\nu) \star (\partial^\star_\lambda \phi) + \frac{i}{2}\theta^{\rho\sigma}\omega^\lambda{}_\rho (\partial^\star_\sigma \partial^\star_\lambda \phi). \end{aligned} \tag{28}$$

Algebra of deformed transformations is undeformed

$$[\hat{\delta}_\omega, \hat{\delta}'_\omega] = \hat{\delta}_{[\omega,\omega']}. \tag{29}$$

However, demanding

$$\delta_\omega\Big(\phi_1 \star \phi_2\Big) \stackrel{\text{def}}{=} -\omega^\lambda{}_\nu x^\nu \partial_\lambda \Big(\phi_1 \star \phi_2\Big), \tag{30}$$

we get the deformed Leibniz rule

$$\begin{aligned} \delta_\omega\Big(\phi_1 \star \phi_2\Big) &= (\delta_\omega \phi_1) \star \phi_2 + \phi_1 \star (\delta_\omega \phi_2) \\ &\quad - \frac{i}{2}\theta^{\rho\sigma}\Big((\delta_{\partial_\rho \omega}\phi_1)(\partial^\star_\sigma \phi_2) + (\partial^\star_\rho \phi_1)(\delta_{\partial_\sigma \omega}\phi_2)\Big) \end{aligned} \tag{31}$$

where $\delta_{\partial_\rho \omega}\phi_1 = -\omega^\lambda{}_\rho(\partial_\lambda \phi_1)$.

The abstract algebra

$$\hat{\delta}_\omega = -\omega^\lambda{}_\nu \hat{x}^\nu \hat{\partial}_\lambda + \frac{i}{2}\theta^{\rho\sigma}\omega^\lambda{}_\rho \hat{\partial}_\lambda \hat{\partial}_\sigma \tag{32}$$

consistent with (2) relations

$$\begin{aligned} \hat{\delta}_\omega \Big([\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu} \Big) &= \Big([\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu} \Big) \hat{\delta}_\omega \\ + \Big([\hat{x}^\alpha, \hat{x}^\nu] - i\theta^{\alpha\nu} \Big)\omega^\mu{}_\alpha &- \Big([\hat{x}^\alpha, \hat{x}^\mu] - i\theta^{\alpha\mu} \Big)\omega^\nu{}_\alpha. \end{aligned} \tag{33}$$

Transformations close (undeformed) algebra

$$[\hat{\delta}_\omega, \hat{\delta}'_\omega] = \hat{\delta}_{[\omega,\omega']}, \tag{34}$$

comultiplication is deformed

$$\begin{aligned} \Delta\hat{\delta}_\omega &= \hat{\delta}_\omega \otimes 1 + 1 \otimes \hat{\delta}_\omega \\ &\quad + \frac{i}{2}\theta^{\rho\sigma}\Big(\omega^\lambda{}_\rho \hat{\partial}_\lambda \otimes \hat{\partial}_\sigma + \hat{\partial}_\rho \otimes \omega^\lambda{}_\sigma \hat{\partial}_\lambda \Big). \end{aligned} \tag{35}$$

Result was also found by Chaichian et al., hep-th/0408069 and by Koch et al., hep-th/0409012.

From (35) one sees that deformed Lorentz transformations do not close bialgebra themselves but derivatives have to be included as well.

Transformation law of a derivative

$$[\hat{\delta}_\omega, \hat{\partial}_\rho] = \omega^\lambda{}_\rho \hat{\partial}_\lambda. \tag{36}$$

θ-deformed Poincaré bialgebra

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0, \quad [\hat{\delta}_\omega, \hat{\partial}_\rho] = \omega_\rho{}^\mu \hat{\partial}_\mu,$$

$$[\hat{\delta}_\omega, \hat{\delta}'_\omega] = \hat{\delta}_{[\omega,\omega']},$$

$$\Delta\hat{\partial}_\rho = \hat{\partial}_\rho \otimes 1 + 1 \otimes \hat{\partial}_\rho,$$

$$\begin{aligned}\Delta\hat{\delta}_\omega &= \hat{\delta}_\omega \otimes 1 + 1 \otimes \hat{\delta}_\omega \\ &\quad + \frac{i}{2}\theta^{\rho\sigma}\Big(\omega^\lambda{}_\rho \hat{\partial}_\lambda \otimes \hat{\partial}_\sigma + \hat{\partial}_\rho \otimes \omega^\lambda{}_\sigma \hat{\partial}_\lambda\Big).\end{aligned}$$

Fields

Scalar field

$$\begin{aligned}\delta_\omega \phi &= -(X_\omega \star \phi) \\ &= -(\omega^\lambda{}_\nu x^\nu) \star (\partial_\lambda \phi) + \frac{i}{2}\theta^{\rho\sigma}\omega^\lambda{}_\rho \partial_\sigma \partial_\lambda \phi\end{aligned} \tag{37}$$

Covariant vector field

$$\begin{aligned}\delta_\omega V_\mu &= -(X^\star_\omega \star V_\mu) - (X^\star_{(\omega^\lambda{}_\mu)} \star V_\lambda) \\ &= -(\omega^\lambda{}_\nu x^\nu) \star (\partial_\lambda V_\mu) + \frac{i}{2}\theta^{\rho\sigma}\omega^\lambda{}_\rho(\partial_\sigma \partial_\lambda V_\mu) - \omega^\lambda{}_\mu V_\lambda.\end{aligned} \tag{38}$$

Contravariant vector field

$$\begin{aligned}\delta_\omega V^\mu &= -(X^\star_\omega \star V^\mu) + (X^\star_{(\omega^\mu{}_\lambda)} \star V^\lambda) \\ &= -(\omega^\lambda{}_\nu x^\nu) \star (\partial_\lambda V^\mu) + \frac{i}{2}\theta^{\rho\sigma}\omega^\lambda{}_\rho(\partial_\sigma \partial_\lambda V^\mu) + \omega^\mu{}_\lambda V^\lambda.\end{aligned} \tag{39}$$

Tensor fields analogously. . .

$$\begin{aligned}\delta_\omega(\phi_1 \star \phi_2) &= -\big(X_\omega \star (\phi_1 \star \phi_2)\big), \\ \delta_\omega(V_\mu \star \phi) &= -\big(X_\omega \star (V_\mu \star \phi)\big) - \big(X^\star_{(\omega^\lambda{}_\mu)} \star (V_\lambda \star \phi)\big) \\ &\dots\end{aligned}$$

because of (31).

Application

or an example of the action that is invariant under the deformed Lorentz transformations

Lagrangian for ϕ^3 theory

$$\mathcal{L} = \frac{1}{2}(\partial^\star_\mu \phi) \star (\partial^{\star\mu}\phi) - \frac{m^2}{2}\phi \star \phi - \lambda\phi \star \phi \star \phi \tag{40}$$

transforms covariantly because of (31)

$$\delta_\omega \mathcal{L} = -(X^\star_\omega \star \mathcal{L}) = -\omega^\lambda{}_\nu x^\nu (\partial_\lambda \mathcal{L}). \tag{41}$$

One also needs an integral with the cyclic property

$$\int \mathrm{d}^4x\, \phi_1 \star \phi_2 = \int \mathrm{d}^4x\, \phi_2 \star \phi_1 = \int \mathrm{d}^4x\, \phi_1\phi_2. \tag{42}$$

The action

$$\begin{aligned} S &= \int \mathrm{d}^4x\, \mathcal{L} \qquad (43)\\ &= \int \mathrm{d}^4x \left(\frac{1}{2}(\partial^\star_\mu \phi) \star (\partial^{\star\mu}\phi) - \frac{m^2}{2}\phi \star \phi - \lambda\phi \star \phi \star \phi\right) \end{aligned}$$

is invariant under deformed Lorentz transformations.

Variational principle

$$\frac{\delta}{\delta g(x)} \int \mathsf{d}^4 x\, f \star g \star h \;=\; \frac{\delta}{\delta g(x)} \int \mathsf{d}^4 x\, g(h \star f)$$
$$= \; h \star f \qquad (44)$$

gives the equation of motion

$$(\partial^{\star\mu} \partial^{\star}_{\mu} \phi) + m^2 \phi + 3\lambda \phi \star \phi = 0. \qquad (45)$$

Expanding the action ($\star$-products) and then varying with the respect to the field ϕ gives the same result.

Conservation law (Nöther's theorem) not clear at the moment; energy-momentum tensor exists, it is conserved but not symmetric; work in progress.

Summary and outlook

- it is possible to construct deformed Lorentz symmetry for the θ-deformed space using the method of inverting the $\star$-product;
  deformed symmetry is deformation of classical Lorentz symmetry;
  algebra is undeformed, but comultiplication (Leibniz rules) changes

- tensor calculus established

- construction of invariant actions, analysis of conserved quantities in progress

- using the same method one deforms the classical diffeomorphism algebra and on that basis constructs a deformed theory of gravity, talk of Frank Meyer