# Fuzzy Newtonian Gravity

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# General introduction and motivation

Physics:

Practical physics: introduce a cut-off $\Lambda$; points ‘fuzzy’ to order $\Lambda^{-1}$

‘Fundamental’ Physics: replace points by ‘Planck cells’; no UV divergences

Solid-state analogy: Coordinates as order parameters; course-graining; screening

Related things: random lattices, qantum nets, twistors, Sakarov’s induced gravity, Wheeler’s graviton as phonon *et cætera*

# Noncommutative geometry and gravity

Replace (Minkowski) coordinates $\tilde{x}^\mu$ by generators $x^\mu$ of a noncommutative algebra $\mathcal{A}_{\kbar}$ with

$$[x^\mu, x^\nu] = i\kbar J^{\mu\nu}, \qquad \kbar \simeq \mu_P^{-2} = G\hbar$$

'Heisenberg' uncertainty relations: $\Lambda^2 \kbar \lesssim 1$

Fuzzy space-time: cells of volume $\simeq (2\pi\kbar)^2$

In the limit $\mu_P \to \infty$: $x^\mu \to z\,\tilde{x}^\mu$

Representation: $x^\mu$ become unbounded hermitian operators on some Hilbert space

The whole idea is contained in the diagram

$$\begin{array}{ccc} \mathcal{A}_{\kbar} & \Longleftarrow & \Omega^*(\mathcal{A}_{\kbar}) \\ \Downarrow & & \Uparrow \\ \text{Cut-off} & & \text{Gravity} \end{array}$$

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The classical Cartan formalism

Special case of 'parallelizable' geometries with

$\mathcal{A} = \mathcal{C}^\infty(V)$: $\Omega^*(V)$: DeRham; $\Omega^1(V)$ free

Frame: $\tilde\theta^i(\tilde e_j) = \delta^i_j$, $\tilde e_i = \tilde e^r_i \tilde\partial_r$

Spin connection: $\tilde d\tilde\theta^i = -\frac{1}{2}\tilde C^i{}_{jk}\tilde\theta^j\tilde\theta^k$

Commutation relations: $[\tilde e_j, \tilde e_k] = \tilde C^i{}_{jk}\tilde e_i$

Gauge condition: $\tilde e_i \tilde C^i{}_{jk} = 0$

## Petrov's classification

In general the manifold has a Weyl tensor $R_{ijkl}$ and a symplectic stucture: $K_{ij}$.

The tensor $K_{ij}$ has 2 principle nul directions $\xi^i$ and can be of type $N$ or $I$

The Weyl tensor has 4 principle null directions $\zeta^i$ and can be of type $N$, $D$, $II$ or $I$

# The NC Cartan formalism

Find an algebra $\mathcal{A}$ with generators $p_i$ and a morphism

$$\tilde{e}_j \mapsto p_j$$

with perhaps a central extension.

## An example

$V = \mathbb{R}^4$ with $\tilde{\theta}^i = d\tilde{x}^i$ and $\tilde{e}_j = \tilde{\partial}_j$

Commutation relations: $[\tilde{e}_j, \tilde{e}_k] = 0$

$$\tilde{e}_j \mapsto p_j$$

Central extension: $[p_j, p_k] = K_{jk}$

Write $p_i = -K_{ij}x^j$

Momentum generators: $p_i$
Position generators: $x^j$
Fourier transform.

Special case of ‘parallelizable’ geometries with

$\mathcal{A}$: $*$-algebra; $\Omega^*(\mathcal{A})$: calculus; $\Omega^1(\mathcal{A})$ free

Frame: $\theta^i(e_j) = \delta^i_j, \qquad e_i = \mathrm{ad}\, p_i$

Dictionary:

$$\begin{aligned} p_i &\mapsto \frac{1}{i\hbar} p_\alpha \\ \theta^i &\mapsto dx^\alpha \\ \theta^i(e_j) &\mapsto \frac{1}{i\hbar}[p_\beta, x^\alpha] \end{aligned}$$

Position ‘space’ and $p$ - ‘space’ are one ‘space’

Dimension $= n =$ rank $\Omega^1(\mathcal{A})$

Reality: $(p_i)^* = -p_i, \qquad (\theta^i)^* = \theta^i$

Constraint equation:

$$[p_k, p_l] = K_{kl} + F^i{}_{kl} p_i - 2i\hbar p_i p_j Q^{ij}{}_{kl}$$

or:

$$[p_k, p_l] = K_{kl} + \frac{1}{2}(F^i{}_{kl} + C^i{}_{kl}) p_i$$

Dictionary:

$$\text{Constraint equation} \quad \mapsto \quad [p_\alpha, p_\beta] = 0$$

According to the general formulae

$$C^\alpha{}_{\beta\gamma} = -2Q^{(\alpha\delta)}{}_{\beta\gamma} p_\delta.$$

It follows therefore that

$$[x^\mu, C^\alpha{}_{\beta\gamma}] = 2P^{(\alpha\delta)}{}_{\beta\gamma} [p_\delta, x^\mu].$$

Both sides vanish in the classical limit.

# Linear approximation

$$f\theta^\alpha = \theta^\alpha f, \qquad \theta^\alpha = \theta^\alpha_\mu dx^\mu$$

$$[x^\lambda, dx^\mu] = i\hbar J^{\lambda\beta} e_\beta e^\mu_\alpha \theta^\alpha$$

$$i\hbar dJ^{\lambda\mu} = i\hbar J^{[\lambda\nu} \partial_\nu e^{\mu]}_\alpha \theta^\alpha$$

That is,

$$dJ^{\lambda\mu} = J^{[\lambda\nu} \partial_\nu e^{\mu]}_\alpha \theta^\alpha + o(i\hbar)$$

From the limiting conditions

$$dJ^{\lambda\mu} = o(i\hbar)$$

we see that the integrability condition for the equation for $J^{\lambda\mu}$ is

$$dJ^{[\lambda\nu} \partial_\nu e^{\mu]}_\alpha \theta^\alpha = o(\hbar^2)$$

Equivalently one can write

$$J^{[\lambda\nu}d\left(\partial_\nu e^{\mu]}_\alpha\theta^\alpha\right) = o(\hbar)$$

With the frame components

$$J^{\lambda\mu} = J^{\alpha\beta}e^\lambda_\alpha e^\pi_\beta$$

we write

$$dJ^{\lambda\mu} = dJ^{\alpha\beta}e^\lambda_\alpha e^\pi_\beta + J^{[\lambda\beta}e_\alpha e^{\mu]}_\beta\theta^\alpha$$

Therefore

$$dJ^{\alpha\beta} + J^{\gamma[\alpha}C^{\beta]}{}_{\gamma\delta}\theta^\delta = o(\hbar^2).$$

This equation has the integrability condition

$$d\left(J^{\gamma[\alpha}C^{\beta]}{}_{\gamma\delta}\theta^\delta\right) = o(\hbar).$$

Therefore

$$J^{\gamma[\alpha} d\left(C^{\beta]}{}_{\gamma\delta}\theta^{\delta}\right) = o(\hbar).$$

In terms of the dual

$$dJ^{*}_{\alpha\beta} + \tfrac{1}{2}C^{\gamma}{}_{\gamma\delta}\theta^{\delta}J^{*}_{\alpha\beta} + \tfrac{1}{2}J^{*}_{\gamma[\alpha}C^{\gamma}{}_{\beta]\delta}\theta^{\delta} = 0.$$

the integrability condition is

$$J^{*}_{[\alpha\beta}e_{\gamma]}C^{\beta}{}_{\zeta\eta} = 0$$

Introduce $\phi$ solution to the equations

$$d\log\phi = C^{\gamma}{}_{\gamma\delta}\theta^{\delta}$$

Then the equations for the dual can be written

$$d(\sqrt{\phi}J^{*}_{\alpha\beta}) + \tfrac{1}{2}\sqrt{\phi}J^{*}_{\gamma[\alpha}C^{\gamma}{}_{\beta]\delta}\theta^{\delta} = 0.$$

# Newtonian approximation

Frame:

$$\theta^0 = (1 - \tfrac{1}{2}\phi)dt, \qquad \theta^i = (1 + \tfrac{1}{2}\phi)dx^i,$$

Notation:

$$\phi = \phi(x^i), \quad A_i = \tfrac{1}{2}[p_i, \phi], \quad A = A_i\theta^i, \quad A = \tfrac{1}{2}d\phi$$

Retain only linear terms:

$$d\theta^0 = -A\theta^0, \qquad d\theta^i = A\theta^i$$

$$C^0{}_{0i} = -A_i, \qquad C^i{}_{jk} = -A_{[j}\delta^i_{k]}$$

Connection

$$\omega^i{}_j = A^i\theta_j - A_j\theta^i, \qquad \omega^0{}_i = -A_i\theta^0$$

Duality:

$$[p_0, t] = 1 + \tfrac{1}{2}\phi, \qquad [p_i, x^j] = (1 - \tfrac{1}{2}\phi)\delta^j_i$$

Ansatz: $A_i = i\epsilon\alpha p_i + i\epsilon\beta_i p_0$.

Algebra relations:

$$[p_0, p_i] = \frac{1}{4} i\epsilon\alpha(p_i p_0 + p_0 p_i) + \frac{1}{2} i\epsilon\beta_i p_0^2 + K_{0i},$$

$$[p_j, p_k] = -\frac{1}{4} i\epsilon\alpha\beta_{[j}(p_0 p_{k]} + p_{k]} p_0) + K_{jk}$$

Jacobi anomaly: $A_J^\alpha$: vanishes if

$$K_{\alpha\beta} = 0$$

Position-algebra commutators

$$dJ^{\mu\nu} = [dx^\mu, x^\nu] - [dx^\nu, x^\mu]$$

We conclude that the commutators are determined by the differential equations

$$dJ^{ij} = A_k J^{k[i} \theta^{j]} \tag{1}$$

$$dJ^{0j} = A_i J^{ij} \theta^0 + A_i J^{i0} \theta^j \tag{2}$$