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Reductions and real Hamiltonian forms of affine Toda field theories

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Plan of the talk:

1. Introduction
2. The reduction group
3. Real forms of semi-simple Lie algebras
4. Spectral properties of the Lax operator
5. The real hamiltonian forms of ATFT
6. Example
7. Conclusions and open problems

1. Introduction

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- It allows Lax representation [Flaschka]:

$$[L, M] = 0$$

where L and M are first order ordinary differential operators whose potentials take values in $\mathfrak{g}$:

$$L\psi \equiv \left(i\frac{d}{dx} - iq_x(x,t) - \lambda J_0 \right) \psi(x,t,\lambda) = 0,$$

$$M\psi \equiv \left(i\frac{d}{dt} - \frac{1}{\lambda} I(x,t) \right) \psi(x,t,\lambda) = 0.$$

- Here $q(x,t) \in \mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, $\vec{q}(x,t) = (q_1, \ldots, q_r)$ is its dual r-component vector, $r = \operatorname{rank} \mathfrak{g}$, and

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- The equations of motion for the Affine Toda field theories are

$$\frac{\partial^2 \vec{q}}{\partial x \partial t} = \sum_{j=0}^{r} n_j \alpha_j \mathrm{e}^{-(\alpha_j, \vec{q})}$$

where n_j are the minimal positive integers s.t. $\sum_{j=0}^{r} n_j \alpha_j = 0$.

- The Lax representations of the ATFT models discussed in the literature [Mikhailov, Olshanetskii, Perelomov; Olive, Turok; Khasgir, Sasaki; Evans, Madsen] are related mostly to the normal real form of the Lie algebra $\mathfrak{g}$.

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- AIMS:

  1) to generalize the ATFT to complex-valued fields;

  2) to describe the family of RHF of these ATFT models;

  3) to construct new inequivalent RHF's of the ATFT's generalizing the results of [Gerdjikov, Kyuldjiev, Marmo, Vilasi] to $1+1$-dimensional systems.

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- The invariance condition has the form:

$$C_k(U(x,t,\kappa_k(\lambda))) = U(x,t,\lambda),$$
$$C_k(V(x,t,\kappa_k(\lambda))) = V(x,t,\lambda),$$

where

$$U(x,t,\lambda) = -iq_x(x,t) - \lambda J_0 \qquad V(x,t,\lambda) = -\frac{1}{\lambda} I(x,t).$$

Here C_k are automorphisms of finite order of $\mathfrak{g}$, i.e. $C_1^h = C_2^2 = (C_1C_2)^2 = \mathbb{1}$ while $\kappa_k(\lambda)$ are conformal mappings of the complex λ-plane.

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♦ These algebraic constraint are automatically compatible with the evolution.

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- Commutation relations for the Cartan–Weyl basis

$$[h_k, E_\alpha] = (\alpha, e_k)E_\alpha, \quad [E_\alpha, E_{-\alpha}] = H_\alpha,$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta}E_{\alpha+\beta} & \text{for } \alpha+\beta \in \Delta \\ 0 & \text{for } \alpha+\beta \notin \Delta \cup \{0\}. \end{cases}$$

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- Real forms: $X \in \mathfrak{g}^{\mathbb{R}}$ if $X \in \mathfrak{g}$ and (see e.g. [Helgasson]):

$$\sigma(\theta(X)) \equiv \theta(\sigma(X)) = X, \qquad \theta(X) = -X^\dagger$$

where σ is an involutive Cartan automorphism: $\sigma^2 = \mathbb{1}$.

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2) If $\sigma(E_\alpha) = -E_\alpha$, the orbit of σ consist of only one element;

3) If $\sigma(E_\alpha) = \varepsilon E_{-\beta}$, $\alpha \neq \beta > 0$ and $\varepsilon = \pm 1$ then $\{\alpha, \beta\}$ is a two-element orbit of σ.

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- The Caudrey–Beals–Coifman systems

$$\tilde{L}m \equiv i\frac{dm}{dx} + iq_x m(x,t,\lambda) - \lambda[J_0, m(x,t,\lambda)] = 0,$$

where $m(x,t,\lambda) = \psi(x,t,\lambda)e^{iJ_0 x\lambda}$. Combining the ideas of [Gerdjikov, Yanovski] with the symmetries of the potential of $L(\lambda)$ one can construct a set of $2h$ fundamental analytic solutions (FAS) $m_\nu(x,t,\lambda)$ of the last eqn. and prove that:

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▶ $m_\nu(x,t,\lambda)$ is analytic with respect to λ in the sector

$\Omega_\nu : (\nu - 1)\pi/h \leq \arg\lambda \leq \nu\pi/h$

satisfying $\lim_{\lambda\to\infty} m_\nu(x,t,\lambda) = \mathbb{1};$

- to each l_ν one relates a subalgebra $\mathfrak{g}_\nu \subset \mathfrak{g}$ such that $\mathfrak{g}_\nu \cap \mathfrak{g}_\mu = \emptyset$ for $\nu \neq \mu \mod (h)$ and $\cup_{\nu=1}^{h} \mathfrak{g}_\nu = \mathfrak{g}$. The symmetry ensure that each of the subalgebras $\mathfrak{g}_\nu$ is a direct sum of $sl(2)$-subalgebras;

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- on Σ the FAS $m_\nu(x,t,\lambda)$ satisfy

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\begin{aligned}
& m_\nu(x,t,\lambda) = m_{\nu-1}(x,t,\lambda) G_\nu(x,t,\lambda), \\
& \lambda \in l_\nu, \\
& G_\nu(x,t,\lambda) = e^{-i(\lambda J_0 x + f(\lambda))t} G_{0,\nu}(\lambda) e^{i(\lambda J_0 x + f(\lambda))t} \in \mathcal{G}_\nu,
\end{aligned}
$$

where $\mathcal{G}_\nu$ is the subgroup with Lie algebra $\mathfrak{g}_\nu$ and $f(\lambda)$ is determined by the dispersion law of the NLEE:

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$f(\lambda) = \sum_{k=0}^{r} E_{-\alpha_k}/\lambda$;
- the FAS satisfy:

$$\widetilde{C}_1(m_\nu(x,t,\omega\lambda)) = m_{\nu-2}(x,t,\lambda), \qquad \lambda \in l_{\nu-2},$$

where $\widetilde{C}_1$ is equivalent to the Coxeter automorphism:

$$\widetilde{C}_1(X) \equiv C_1^{-1} X C_1,$$

$$C_1 = \exp(\frac{2\pi i}{h} H_\rho), \qquad \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha;$$

obviously $C_1^h = 1\!\!1$ and $\widetilde{C}_1(J_0) = \omega^{-1} J_0$;

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obviously $C_1^h = 1\!\!1$ and $\widetilde{C}_1(J_0) = \omega^{-1} J_0$;

- the FAS $m_\nu(x,t,\lambda)$ satisfy one of the following two involutions:

$$\widetilde{C}_2(m_\nu(x,t,\lambda^*))^\dagger = C_2(m_{2h-\nu+2}^{-1}(x,t,\lambda)),$$

$$(m_\nu(x,t,-\lambda^*))^* = m_{h-\nu+2}(x,t,\lambda).$$

where C_2, $C_2^2 = 1\!\!1$ is conveniently chosen Weyl group element.

- These relations lead to the following constraints for the sewing functions $G_{0,\nu}(\lambda)$ and the minimal set of scattering data:

$$\widetilde{C}_1(G_{0,\nu}(\omega\lambda)) = G_{0,\nu-2}(\lambda),$$
$$\widetilde{C}_2(G^{\dagger}_{0,\nu}(\lambda^*)) = G^{-1}_{0,2h-\nu+2}(\lambda),$$
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- If L has no discrete eigenvalues then the minimal set of scattering data is provided by the coefficients of $G_{0,1}(\lambda), \lambda \in l_1$ and $G_{0,2}(\lambda), \lambda \in l_2$.

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▲ All other sewing functions $G_{0,\nu}(\lambda)$ can be determined from them.

5. The real Hamiltonian forms of ATFT

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- The ATFT can be written down as an infinite-dimensional Hamiltonian system as follows:

$$\frac{dq_k}{dt} = \{q_k, H\}, \qquad \frac{dp_k}{dt} = \{p_k, H\},$$

$$H_{\mathrm{ATFT}} = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}(\vec{p}(x,t), \vec{p}(x,t)) + \sum_{k=0}^{r} n_k e^{-(\vec{q}(x,t),\alpha_k)} \right),$$

where $\vec{p} = d\vec{q}/dx$ and $\vec{q}$ are the canonical momenta and coordinates satisfying canonical Poisson brackets:

$$\{q_k(x,t), p_j(y,t)\} = \delta_{jk}\delta(x-y).$$

- Next we introduce an involution $\mathcal{C}$ acting on the phase space $\mathcal{M} \equiv \{q_k(x), p_k(x)\}_{k=1}^{n}$ and satisfying:

$$1) \qquad \mathcal{C}(F(p_k, q_k)) = F(\mathcal{C}(p_k), \mathcal{C}(q_k)),$$

$$2) \qquad \mathcal{C}\left(\{F(p_k, q_k), G(p_k, q_k)\}\right) = \{\mathcal{C}(F), \mathcal{C}(G)\},$$

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- The complex ATFT (CATFT) can be written down as standard Hamiltonian system with twice as many fields $\vec{q}_a(x,t)$, $\vec{p}_a(x,t)$, $a = 0, 1$ (in what follows we will skip the x and t dependence):

$$\vec{p}^{\mathbb{C}}(x,t) = \vec{p}^{0}(x,t) + i\vec{p}^{1}(x,t), \quad \vec{q}^{\mathbb{C}}(x,t) = \vec{q}^{0}(x,t) + i\vec{q}^{1}(x,t),$$

$$\{q_k^0(x,t), p_j^0(y,t)\} = -\{q_k^1(x,t), p_j^1(y,t)\} = \delta_{kj}\delta(x-y).$$

- The densities of the corresponding Hamiltonian and symplectic form equal

$$\begin{aligned} \mathcal{H}^{\mathbb{C}}_{\mathrm{ATFT}}(x,t) &\equiv \mathrm{Re}\,\mathcal{H}_{\mathrm{ATFT}}(\vec{p}^{\,0} + i\vec{p}^{\,1}, \vec{q}^{\,0} + i\vec{q}^{\,1}) \\ &= \frac{1}{2}(\vec{p}^{\,0}, \vec{p}^{\,0}) - \frac{1}{2}(\vec{p}^{\,1}, \vec{p}^{\,1}) \\ &+ \sum_{k=0}^{r} n_k e^{-(\vec{q}^{\,0}, \alpha_k)} \cos((\vec{q}^{\,1}, \alpha_k)), \\ \omega^{\mathbb{C}}(x,t) &= (d\vec{p}^{\,0} \wedge i d\vec{q}^{\,0} - d\vec{p}^{\,1} \wedge d\vec{q}^{\,1}). \end{aligned}$$

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  ▶ The involution $\mathcal{C}$ splits the initial real phase space $\mathcal{M}$ into a direct sum

$$\mathcal{M} \equiv \mathcal{M}_0 \oplus \mathcal{M}_1$$

defined by:

$\mathcal{C}(X) = X$, for any $X \in \mathcal{M}_0$ and

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$\mathcal{M}^{\mathbb{C}} \equiv \mathcal{M}^{\mathbb{C}}_{+} \oplus \mathcal{M}^{\mathbb{C}}_{-}$,

where

$$\mathcal{M}^{\mathbb{C}}_{+} = \mathcal{M}_0 \oplus i\mathcal{M}_1, \qquad \mathcal{M}^{\mathbb{C}}_{-} = i\mathcal{M}_0 \oplus \mathcal{M}_1,$$

defined by:

$\mathcal{C}(X) = X$, for any $X \in \mathcal{M}_0$ and

$\mathcal{C}(Y) = -Y$, for any $Y \in \mathcal{M}_1$.

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- The automorphism $\mathcal{C}$ is dual to an automorphism $\mathcal{C}^{\#}$ of the corresponding Lax pair and the Lie algebra $\mathfrak{g}$. In fact $\tilde{\mathcal{C}}^{\#} = -\mathcal{C}^{\#}(X^{\dagger})$ is a Cartan involution of $\mathfrak{g}$ and therefore the Lax pair of the RHF is related to a real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$.

- Thus to each involution $\mathcal{C}$ satisfying 1) - 3) one can relate a RHF of the ATFT. Due to the condition 3) $\mathcal{C}$ must preserve the system of admissible roots of $\mathfrak{g}$;

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- such involutions can be constructed from the $\mathbb{Z}_2$-symmetries of the extended Dynkin diagrams of $\mathfrak{g}$ [Khasgir, Sasaki].

6. Example

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- We choose $\mathfrak{g} \simeq A_{2r+1}$ and fix up the involution $\mathcal{C}$ by:

$$\mathcal{C}(q_k) = -q_{2r+2-k}, \qquad \mathcal{C}(p_k) = -p_{2r+2-k},$$
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▶ The coordinates in $\mathcal{M}_\pm$ are given by:

$$\begin{aligned} q_k^\pm &= \frac{1}{\sqrt{2}}(q_k \mp q_{2r+2-k}), \\ p_k^\pm &= \frac{1}{\sqrt{2}}(p_k \mp p_{2r+2-k}), \\ q_{r+1}^- &= q_{r+1}, \qquad p_{r+1}^- = p_{r+1}, \end{aligned}$$

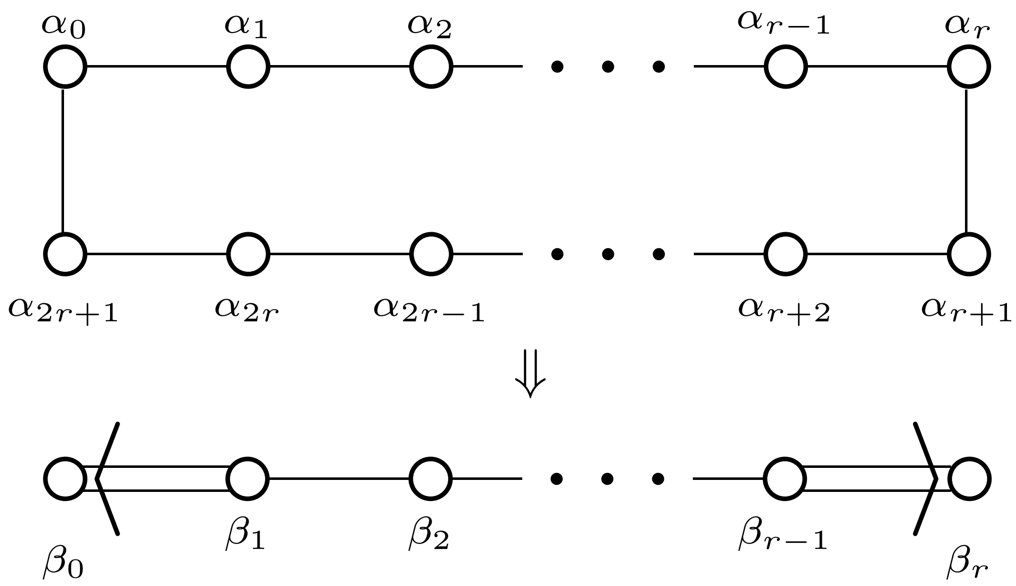
where $k = 1, \ldots, r$, i.e., $\dim \mathcal{M}_+ = 2r$ and $\dim \mathcal{M}_- = 2r + 2$.

- Then the densities $\mathcal{H}^{\mathbb{R}}_{\mathrm{ATFT}}$, $\omega^{\mathbb{R}}_{\mathrm{ATFT}}$ for the RHF of AFTF equal:

$$\mathcal{H}^{\mathbb{R}}_{\mathrm{ATFT}}(x,t) = \frac{1}{2}\sum_{k=1}^{r} p_k^{+\,2} - \frac{1}{2}\sum_{k=1}^{r+1} p_k^{-\,2}$$
$$+2e^{-q^{+}_{r-1}/\sqrt{2}} \cos\left(\frac{q^{-}_{r-1}}{\sqrt{2}} - q^{-}_{r+1}\right)$$
$$+\sum_{k=1}^{r-1} 2e^{(q^{+}_{k+1}-q^{+}_{k})/\sqrt{2}} \cos\left(\frac{q^{-}_{k+1} - q^{-}_{k}}{\sqrt{2}}\right)$$
$$+2e^{q^{+}_{1}/\sqrt{2}} \cos\left(\frac{q^{-}_{1}}{\sqrt{2}} - q^{-}_{r+1}\right)$$

$$\omega^{\mathbb{R}}_{\mathrm{ATFT}}(x,t) = \sum_{k=1}^{r} dp_k^{+} \wedge dq_k^{+} - \sum_{k=1}^{r+1} dp_k^{-} \wedge dq_k^{-}$$

where $\vec{p}^{\pm}_k(x,t) = d\vec{q}^{\pm}_k/dx$.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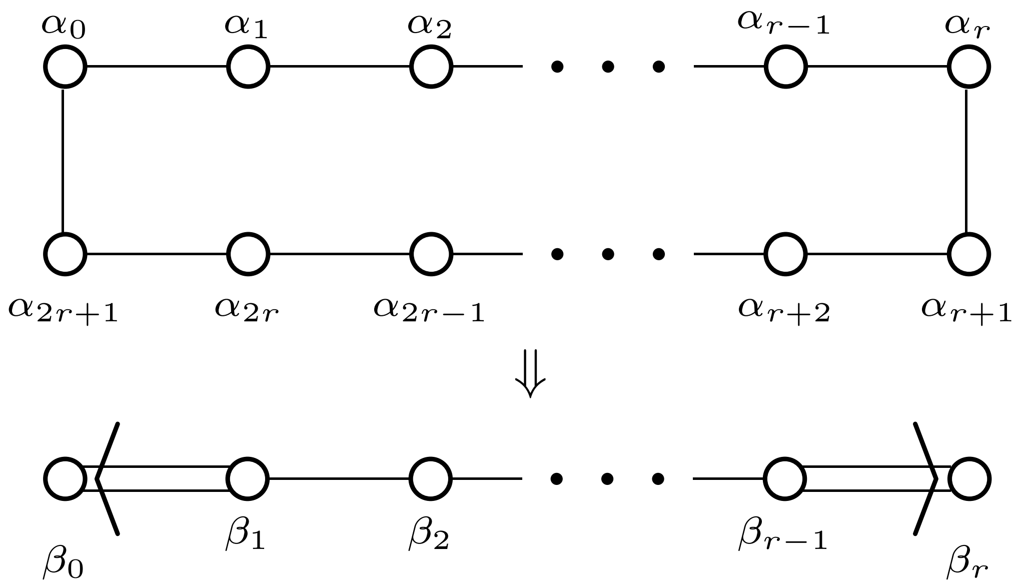
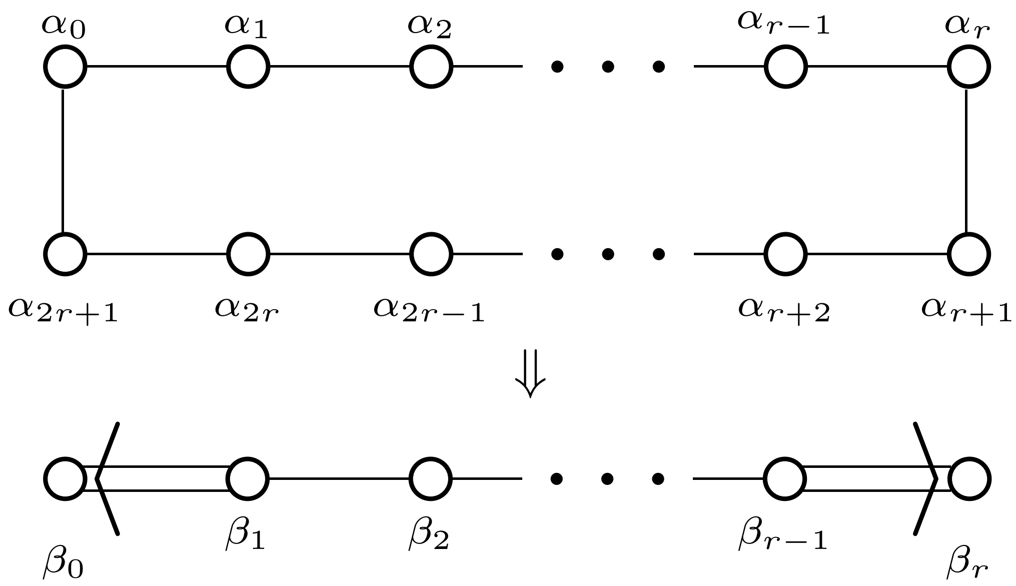


► This is a generalization of the results of [Khasgir, Sasaki] for the reduced ATFT related to the Kac-Moody algebra $D^{(2)}_{r+1}$; the latter are obtained if we put $q^-_{r+1} = 0$ and $p^-_{r+1} = 0$.



- This is a generalization of the results of [Khasgir, Sasaki] for the reduced ATFT related to the Kac-Moody algebra $D_{r+1}^{(2)}$; the latter are obtained if we put $q_{r+1}^{-} = 0$ and $p_{r+1}^{-} = 0$.
- Note the additional trigonometric in the Hamiltonian in addition to the standard exponential ones.

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  ▶ Imposing the reduction conditions on these parameters one can obtain the soliton solutions of the RHF of the ATFT.

  ▶ On the example for the Toda chain we found that going from one RHF to another may change qualitatively the dynamics of the system. Part of the formerly non-compact trajectories may become compact and vice-versa.

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  - ► To apply this method to other integrable models, as e.g. the $\mathbb{Z}_n$-nonlinear Schrödinger equation.
  - ► To study the asymptotical dynamical regimes of the RHF of ATFT.

Thank you!