

# Towards $B$ -field in $p$ -string theory

Debashis Ghoshal

Harish-Chandra Research Institute  
Allahabad, India

Based on

DG, hep-th/0406259 (in JHEP)

DG & T. Kawano, hep-th/0409311 (in NPB)

Vrjnacka Banja, 2005

## Plan of the talk:

1. Introduction & motivation
2. Brief review of  $p$ -adic string theory
3. Noncommutative  $p$ -soliton
  - +  $p \rightarrow 1$  Limit & BSFT
  - + Comments on multisolitons
4. *To  $B$  or not to  $B$ ?*  
Constant  $B$ -field in the 'worldsheet'
  - + Green's function & formal consequences
  - + Problem with projective invariance
  - + Tachyon scattering amplitudes
5. Summary & some remarks

**Noncommutative deformations** of quantum field theories have been studied during the past few years. The deformed theories exhibit rather interesting perturbative and nonperturbative behaviour.

Local field theory  $\xrightarrow{\text{nc def}}$  **Non-local field theory**

Some (effective) field theories are **nonlocal** to start with. We shall study one such.

It is the **effective field theory** of a scalar field: the **tachyonic scalar** of the so called  **$p$ -adic string theory**.

(Freund & Olson; Freund & Witten; Brekke, Freund, Olson & Witten; Frampton & Okada; Frampton, Okada & Ubricaco; ...)

(Also Volovic; Grossmann; ...)

The  $p$ -adic string theory is an interesting theory. (A sector) of this theory is amenable to exact calculations.

It shares many qualitative properties of usual string theory. For example, one can check the Sen conjectures for the open string tachyon in the exact effective field theory.

Therefore,  $p$ -adic string theory could be a useful guide to difficult questions in (usual) string theory.

However, this requires a better understanding of the  $p$ -adic string itself. We only know some properties of its D-branes in flat spacetime.

What about putting  $p$ -adic strings in non-trivial backgrounds?

For example,

- in **curved spacetime**: from **closed string condensate**;
- in **electromagnetic field**: from **open string background**.

A particularly simple background: **a constant background of the rank two antisymmetric tensor field  $B$** .

It provides a **noncommutative deformation** of the **effective field theory** in spacetime.

(Schomerus; Witten; ...)

***Assumption*** (for the first part):

the **same** happens for  $p$ -adic string theory.

## Abbreviations

$p$ -tachyon : Tachyon of  $p$ -adic string theory

(Cf: *Ptolemy*, *Pterodactyl* and *Ptennisnet*!)

Similarly:

$p$ -string :  $p$ -adic string

$p$ -soliton : Soliton of  $p$ -string theory

`\begin{digression}`

Consider the *rational numbers*  $\mathbb{Q}$ . We are familiar with the *absolute value norm*. Now we will define *different norms* on this field.

1. Fix a *prime number*  $p \in \{2, 3, 5, 7, 11, 13, \dots\}$ ,
2. For any *integer*  $z$ , find the *highest power* of  $p$  that divides it. Call it  $\text{ord}_p z$ .

$$\text{ord}_p \sim \text{logarithm: } \text{ord}_p(z_1 z_2) = \text{ord}_p z_1 + \text{ord}_p z_2.$$

3. For a *rational number*  $q = \frac{z_1}{z_2}$ :  
 $\text{ord}_p q = \text{ord}_p z_1 - \text{ord}_p z_2$ .

4. The *p-adic norm* is:

$$|q|_p = \begin{cases} p^{-\text{ord}_p q}, & q \neq 0 \\ 0, & q = 0 \end{cases}$$

**Examples:**  $|5|_5 = \frac{1}{5}$ ,  $|35|_5 = \frac{1}{5}$ ,  $|250|_5 = \frac{1}{5^3} = \frac{1}{125}$ ,  
 $|6|_5 = 1$ ,  $|\frac{2}{3}|_5 = 1$ ,  $|\frac{2}{5}|_5 = 5$ ,  $|96|_2 = \frac{1}{32}$ .

$|q|_p$  satisfies all the properties of a norm. In fact there is a **stronger** triangle inequality:

$$|q_1 + q_2|_p \leq \max \{ |q_1|_p, |q_2|_p \}$$

This norm is called **non-Archimedean** and leads to many strange properties:

- All triangles are **isosceles**.
- Any point in the **interior** of a disc is a **centre**.
- If  $|q_1|_p < |q_2|_p$ , then  $|n q_1|_p < |q_2|_p$  for any  $n \in \mathbf{Z}$ .

Now recall that the field of real number  $\mathbf{R}$  is obtained from  $\mathbf{Q}$  adding to  $\mathbf{Q}$  the **limit points of Cauchy sequences**. The limit points are determined by the absolute value norm.

**Cauchy completion of  $\mathbf{Q}$  by the  $p$ -adic norm** leads to a **new** field of numbers  $\mathbf{Q}_p$



Any number in  $\mathbf{Q}_p$  can be represented as a power series in  $p$

$$\xi = p^N (\xi_0 + \xi_1 p + \xi_2 p^2 + \dots)$$

where,  $\xi_n = \{0, 1, 2, \dots, p-1\}$ ,  $\xi_0 \neq 0$ .

E.g. in  $\mathbf{Q}_7$ ,  $\frac{1}{2} = 4 + 3.7 + 3.7^2 + 3.7^3 + \dots$ .

This is like the Laurent series expansion.

Elements of  $\mathbf{Q}_p$  with norm at most 1, form a maximal compact subring:

$$\mathbf{Z}_p = \{\xi \in \mathbf{Q}_p : |\xi|_p \leq 1\},$$

the ring of  $p$ -adic integers. Ordinary integers are a dense subset in it.

Integration in  $\mathbb{Q}_p$ : There is *translationally invariant* real valued Haar measure  $dx$ .

Examples:

- Normalization:  $\int_{\mathbb{Z}_p} d\xi = 1$ .

- Gelfand-Graev-Tate gamma-function:

$$\Gamma(s) = \int_{\mathbb{Q}_p} d\xi e^{2\pi i \{\xi\}} |\xi|_p^{s-1} = \frac{1 - p^{s-1}}{1 - p^{-s}}$$

[Cf:  $\Gamma_{\mathbb{R}}(s) = \int_{\mathbb{R}} dx e^{2\pi i \{x\}} |x|^{s-1} = \frac{2 \cos(\pi s/2)}{(2\pi)^s} \Gamma_{\text{Euler}}(s)$ .]

and beta-function:

$$\int_{\mathbb{Q}_p} d\xi |\xi|_p^{x-1} |1 - \xi|_p^{y-1} = \Gamma(x) \Gamma(y) \Gamma(1-x-y)$$

`\end{digression}`

## Review of $p$ -string theory

(à la Brekke, Freund, Olson & Witten (1987))

Recall tree level scattering amplitude of  $N$  on-shell tachyons of momenta  $k_i$  ( $i = 1, \dots, N$ ),  $k_i^2 = 2, \sum k_i = 0$  is:

$$\begin{aligned} \mathcal{A}_N(\{k_i\}) &= \int d\xi_1 \cdots d\xi_{N-3} \\ &\times \prod_{i=1}^{N-3} |\xi_i|^{k_N \cdot k_i} |1 - \xi_i|^{k_{N-1} \cdot k_i} \\ &\times \prod_{1 \leq i < j \leq N-3} |\xi_i - \xi_j|^{k_i \cdot k_j}. \end{aligned}$$

The integrals are over the real line  $\mathbf{R}$  with its translationally invariant measure  $d\xi$ . Integrand involves absolute values of real numbers.

The 4-point amplitude can be computed exactly in terms of Euler beta function. But  $\mathcal{A}_N$  for  $N \geq 5$  cannot be evaluated analytically.

The key observation of *BFW*: all the ingredients of the amplitude  $\mathcal{A}_N$  have *p-adic analogues*.

*Proposal*: Define *p-string theory* as follows:

- replace the absolute value norm  $|\cdot|$  of  $\mathbf{R}$  in the integrand by the *p-adic norm*  $|\cdot|_p$  of  $\mathbf{Q}_p$ :

$$|\cdot| \rightarrow |\cdot|_p,$$

- replace integrals over  $\mathbf{R}$  by integrals over  $\mathbf{Q}_p$  (using the latter's translationally invariant measure).

The consequence of this is amazing:

*All N-point amplitudes* for *p-tachyons* can now be computed *exactly*.

*I.e.*, we know the *effective field theory* of the tachyonic scalar field of *p-string theory* *exactly*.

Dynamics of the  $p$ -tachyon field is summarized by its *spacetime* effective lagrangian:

$$\mathcal{L}^{(p)} = \frac{p^2}{g^2(p-1)} \left[ -\frac{1}{2} \varphi p^{-\frac{1}{2}} \square \varphi + \frac{1}{p+1} \varphi^{p+1} \right].$$

The above is written in terms of

$$\varphi(x) = 1 + \frac{g}{p} T(x),$$

$T$  is the original  $p$ -tachyon.

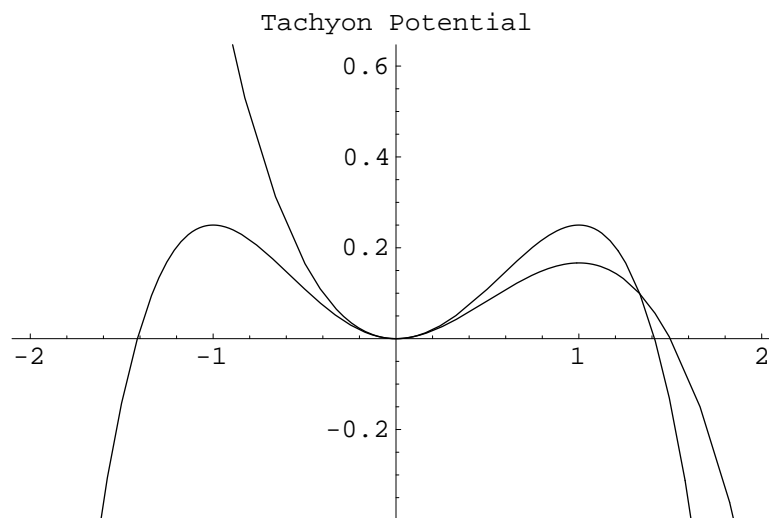
Perturbative vacuum  $T = 0$  correspond to  $\varphi = 1$ .

*Parenthetic remark: Although derived for a prime  $p$ , the spacetime action makes sense for all integer values of  $p$ .*

Kinetic term in is **non-standard** and involves an **infinite** number of derivatives.

Potential has a **local minimum** and **two** (respectively **one**) **local maxima** for **odd** (respectively **even**) integer  $p$ :

### **$p$ -tachyon potential**



Potential always has runaway pathological **singularities**.

Equation of motion:

$$p^{-\frac{1}{2}} \square \varphi = \varphi^p.$$

The *solutions* to these are:

- The *constant* configuration  $\varphi(x) = 1$ .  
This is the vacuum around which we have quantized — a *space-filling D-25-brane*.  
The *tension* of the brane is

$$\mathcal{T}_{25} = -\mathcal{L}(\phi = 1) = \frac{1}{2g^2} \frac{p^2}{p+1}.$$

- The *constant* configuration  $\varphi(x) = 0$ .  
There are *no* *perturbative excitation* around this configuration. Energy vanishes in this configuration.

*Closed string vacuum?*

- *Space(time) dependent* solutions:

The eqn of motion is *separable* in orthogonal coordinates.

$$p^{-\frac{1}{2}} \partial_y^2 f(y) = f^p(y)$$

This has a non-trivial solution:

$$f(y) = p^{1/2p(p-1)} \exp\left(-\frac{p-1}{2p \ln p} y^2\right).$$

Gaussian lump with *correlated amplitude* and spread.

The configurations

$$\varphi(x) = f(x^{m+1}) f(x^{m+2}) \dots f(x^{25}) = F^{(m)}(x_{\perp})$$

is a *solution* to the eqn of motion for  $m = 0, 1, \dots, 24$ .

$$\underbrace{(x^0, x^1, \dots, x^m)}_{x_{\parallel}}, \underbrace{(x^{m+1}, \dots, x^{25})}_{x_{\perp}}$$

$\varphi(x) = F^{(m)}(x_{\perp})$  describes a *lump* with energy *localised* around the hyperplane  $x_{\perp} = 0$ .



This is a solitonic  $m$ -brane.

Its tension (=energy/volume) is

$$\begin{aligned} \mathcal{T}_m &= \frac{p^2}{2g^2(p+1)} \left[ \frac{2\pi p^{2p/(p-1)} \ln p}{p^2 - 1} \right]^{(25-m)/2} \\ &\equiv \frac{1}{2g_m^2} \frac{p^2}{p+1}. \end{aligned}$$

Ratio of tension  $\frac{\mathcal{T}_m}{\mathcal{T}_{m-1}}$  is independent of  $m$ : as in ordinary string theory.

Can study the worldvolume theory on the soliton. A consistent truncation keeping only the tachyon is possible. There are also massless fields corresponding to translation zero modes.

$p$ -Tachyon field behaves according to the conjectures of Sen. (DG & Sen, 2000)

## Noncommutative field theories

Spacetime coordinates do **not** commute:

$$[x^i, x^j] = i\theta\epsilon^{ij}.$$

For simplicity,  $i, j = 1, 2$  and  $\theta = \text{real constant}$ .

In string theory, the **deformation** arises from a **constant** background of the **two-form antisymmetric tensor field**  $B$  along  $x^1$ - $x^2$ .

An **equivalent** description: use **commuting** coordinates, but while multiplying fields, which are dependent on  $x^1$  and  $x^2$ , use the **Moyal star product**

$$f \star g = f(x^1, x^2) \exp\left(\frac{i}{2}\theta\epsilon^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right) g(x^1, x^2),$$

instead of ordinary pointwise multiplication.

Alternatively,  $f, g$  are **operator-valued functions** through the Moyal-Weyl correspondence.

(Commutative) field theories may be deformed by using Moyal star product in the action, equations of motion, etc.

Example: theory of a single scalar field  $\phi$ :

$$\mathcal{L}_{NC} = \frac{1}{2}(\partial_\mu\phi) \star (\partial^\mu\phi) - \mathcal{V}(\star\phi),$$

where,  $\mathcal{V}(\star\phi)$  may be, e.g.,

$$\mathcal{V}(\star\phi) \sim \frac{m^2}{2}\phi\star\phi + \frac{g}{3!}\phi\star\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi + \dots$$

Noncommutative field theories exhibit interesting perturbative and nonperturbative properties.

Noncommutative scalar field theory has soliton solutions in the limit of infinite noncommutativity:  $\theta \rightarrow \infty$ .

(Gopakumar, Minwalla & Strominger; ...)

The simplest of the solitons is a gaussian lump whose width and amplitude are fixed:

$$\phi(x^1, x^2) = 2 \exp\left(-\frac{(x^1)^2 + (x^2)^2}{\theta}\right). \quad (1)$$

More general solutions correspond to the solutions of projector equation:

$$\pi \star \pi \sim \pi.$$

However, these solitons become unstable at finite  $\theta$ :

Higher projectors are unstable and decay to the gaussian lump,

Gaussian lump becomes unstable below a critical value of  $\theta$ .

*These (and other) noncommutative solitons have been used in string (field) theory to understand the dynamics of the tachyon. (Dasgupta, Mukhi & Rajesh; Harvey, Kraus, Larsen & Martinec; ...)*

## Noncommutative deformation of the $p$ -tachyon

Use  $\star$ -product in the  $p$ -tachyon action:

$$\mathcal{L}_{NC}^{(p)} = -\varphi \star p^{-\frac{1}{2}\square} \varphi + \frac{1}{p+1} (\star\varphi)^{p+1},$$

We define the noncommutative  $p$ -tachyon by the above action.

Equation of motion:

$$p^{-\frac{1}{2}\square} \varphi = (\star\varphi)^p.$$

We will be interested only in the part dependent on  $x^1$  and  $x^2$ .

The solutions,  $\varphi = 0, 1$  describing constant configurations, are still solutions of the deformed eom.

There are **nontrivial solutions**: these are **gaussian solitonic lumps**.

**Observe**:  $\star$ -product of gaussian is again a gaussian:

$$Ae^{-a|z|^2} \star B e^{-b|z|^2} = \frac{AB}{1 + ab\theta^2} \exp\left(\frac{a + b}{1 + ab\theta^2}|z|^2\right).$$

Make a gaussian **ansatz**:

$$\varphi(x^1, x^2) = A^2 \exp\left(-a \left[(x^1)^2 + (x^2)^2\right]\right)$$

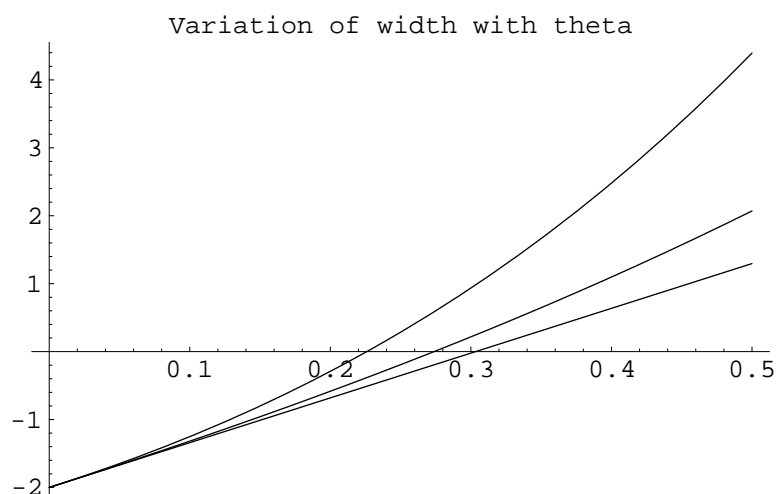
and **equate width**  $a$  and **amplitude**  $A$  on both sides of eom.

Spread  $a$  is determined by a polynomial equation of degree  $p$ :

$$\sum_{i=0}^{\lfloor p/2 \rfloor} \binom{p}{2i} (a\theta)^{2i} - (1 - 2a \ln p) \sum_{i=0}^{\lfloor (p-1)/2 \rfloor} \binom{p}{2i+1} (a\theta)^{2i} = 0,$$

For an odd  $p$ , there is always one real root. For  $p = 2$ , the roots are real, with the positive root being the relevant one.

### The polynomial for $p = 3$



Limiting cases:

- Commutative limit:  $\theta = 0$

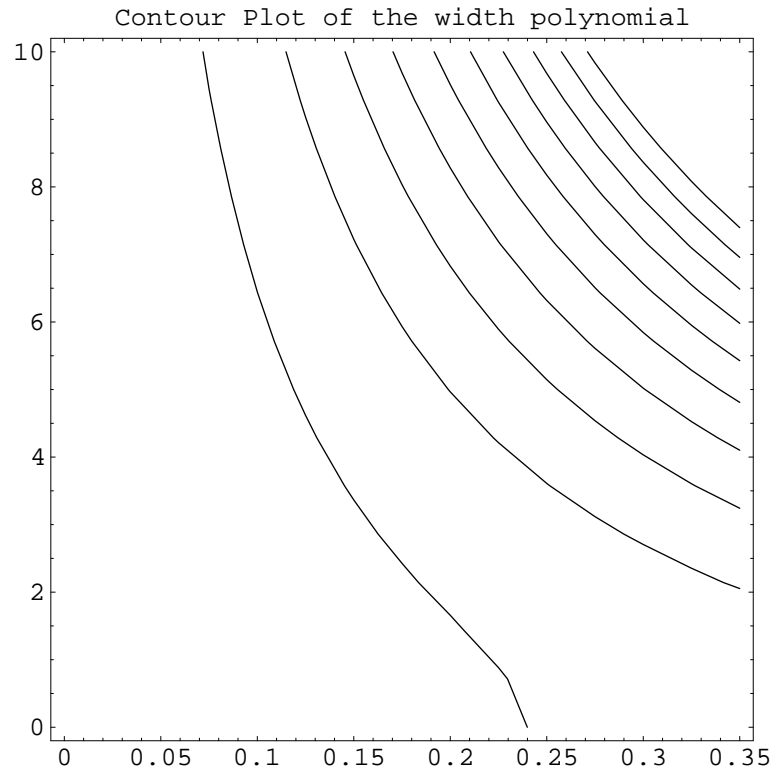
The width of the commutative gaussian lump of  $p$ -string theory  $a = (p - 1)/2p \ln p$  is recovered.

- Large noncommutative limit:  $\theta \rightarrow \infty$

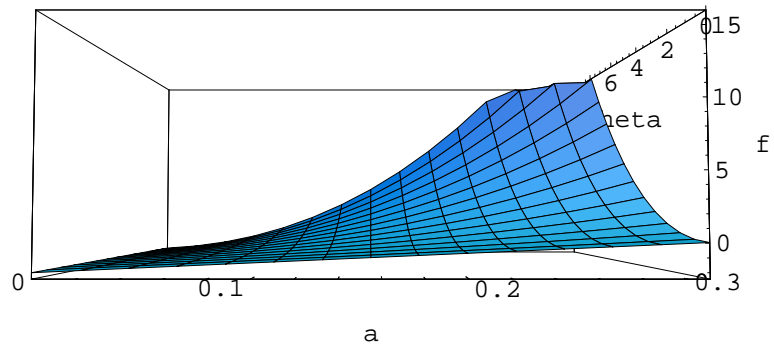
Keeping only the term of highest degree in  $a$  naively is not the right thing. Indeed using  $a \sim 1/\theta$  for large  $\theta$ , this is subdominant. Correct treatment gives  $a = 1/\theta$ .



## Width as a function of $\theta$



## Polynomial as a function of $a$ and $\theta$



Amplitude  $A$  is determined in terms of the width:

$$A = \left[ \sum_{i=0}^{\lfloor (p-1)/2 \rfloor} \binom{p}{2i+1} (a\theta)^{2i} \right]^{1/2(p-1)}.$$

This, again, interpolates between

- Commutative limit ( $\theta = 0$ ):  
 $A = p^{1/2(p-1)},$
- Large noncommutativity limit ( $\theta \rightarrow \infty$ ):  
 $A = \sqrt{2}.$

Summary:

Noncommutative  $p$ -soliton interpolates smoothly between noncommutative soliton and commutative  $p$ -soliton.

## $p \rightarrow 1$ Limit & BSFT

In this limit the lagrangian of  $p$ -string theory reduces to:

$$\mathcal{L}^{(p \rightarrow 1)} = -\frac{1}{2}\varphi \square \varphi - \frac{1}{2}\varphi^2 (\ln \varphi^2 - 1).$$

This is exactly the **boundary string field theory** (BSFT) action for the **tachyon** of ordinary bosonic string, **truncated to two derivatives!** (Gerasimov & Shatashvili; Kutasov, Marino & Moore)

Introduce **noncommutativity** [Cornalba, Okuyama]:

$$\mathcal{L}_{NC}^{(p \rightarrow 1)} = -\frac{1}{2}\varphi \star \square \varphi - \frac{1}{2}\varphi \star \varphi (\ln_{\star}(\varphi \star \varphi) - 1)$$

yields the **equation of motion**:

$$\square \varphi + 2\varphi \star \ln_{\star} \varphi = 0.$$

Solutions:

- Constant configurations  $\varphi = 0, 1$ ,
- Deformation of gaussian lump of BSFT?

Gaussian ansatz: LHS is exactly as in the commutative case.

$$\square\varphi = 4a(a|z|^2 - 1) A^2 \exp(-a|z|^2).$$

RHS is less obvious, since we need the  $\star$ -deformed logarithm of an ordinary exponential.

From the  $n$ -fold  $\star$ -product of the gaussian, we can extrapolate to fractional powers.

For  $g(z) = Ae^{-a|z|^2}$ :

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[ (\star g(z))^{1+\epsilon} - g(z) \right] \\ &= \left[ \left( 2 \ln A - \frac{(a\theta)^2}{1.2} - \frac{(a\theta)^4}{3.4} - \dots \right) \right. \\ & \quad \left. - 2a|z|^2 \left( \frac{1}{2} - \frac{(a\theta)^2}{1.3} - \frac{(a\theta)^4}{3.5} - \dots \right) \right] A^2 e^{-a|z|^2} \end{aligned}$$

Hence, by comparison

$$\begin{aligned} 2a &= \frac{1 - (a\theta)^2}{2a\theta} \ln \left( \frac{1 + a\theta}{1 - a\theta} \right) \\ 2 \ln A &= \frac{1 + a\theta}{2a\theta} \ln(1 + a\theta) - \frac{1 - a\theta}{2a\theta} \ln(1 - a\theta) \end{aligned}$$

Width  $a$  and amplitude  $A$  are determined by *transcendental* equation.

They *interpolate smoothly* between the **BSFT** solution  $a = \frac{1}{2}$ ,  $A = \sqrt{e}$  to the **GMS** solution  $a = \frac{1}{\theta}$ ,  $A = \sqrt{2}$ .

## Multisolitons

Consider

$$f_{w_1 w_2} = \exp\left(-\frac{1}{\theta}(\bar{z} - \bar{w}_1)(z - w_2)\right).$$

The configuration

$$\sum_{i,j=1}^n A_{ij} f_{w_i w_j} = \sum_{i,j} A_{ij} e^{-(\bar{z} - \bar{w}_i)(z - w_j)/\theta}$$

solves the eom  $\pi \star \pi \sim \pi$  at  $\theta \rightarrow \infty$  for specific choices of  $A_{ij}$ .

(Headrick, Gopakumar & Spradlin)

By Moyal-Weyl correspondence

$$f_{w_i w_j} \sim |w_i\rangle\langle w_j|,$$

where,

$$|w\rangle \sim e^{w a^\dagger} |0\rangle$$

is a coherent state. The amplitudes of a multisoliton are

$$\|A_{ij}\| \sim \|\langle w_i | w_j \rangle\|^{-1}.$$

More generally, define:

$$f_{w_i w_j}(a_{ij}, A_{ij}) = A_{ij} \exp \left( -a_{ij}(\bar{z} - \bar{w}_i)(z - w_j) \right).$$

The **LHS** of the **eom** of the  $p$ -tachyon:

$$\begin{aligned} & e^{-\frac{1}{2} \ln p} f_{w_i w_j}(a_{ij}, A_{ij}) \\ &= f_{w_i w_j} \left( \frac{a_{ij}}{1 + 2a_{ij} \ln p}, \frac{A_{ij}}{1 + 2a_{ij} \ln p} \right). \end{aligned}$$

However, at arbitrary  $\theta$ , the **RHS** involves **★-product** of these  $f_{w_i w_j}(a_{ij}, A_{ij})$ . This is given by:

$$f_{w_1 w_2}(a_{12}, A_{12}) \star f_{w_3 w_4}(a_{34}, A_{34}) = f_{\tilde{u} \tilde{v}}(\tilde{a}, \tilde{A}),$$

where, the **modified width**  $\tilde{a}$ , **amplitude**  $\tilde{A}$  and the **centres**  $\tilde{u}$  and  $\tilde{v}$  are given by:

$$\begin{aligned}
\tilde{a} &= \frac{a_{12} + a_{34}}{1 + \theta^2 a_{12} a_{34}} \\
\tilde{A} &= \frac{A_{12} + A_{34}}{1 + \theta^2 a_{12} a_{34}} \\
&\quad \times \exp \left\{ -\frac{a_{12} a_{34}}{a_{12} + a_{34}} (\bar{w}_1 - \bar{w}_3)(w_2 - w_4) \right\} \\
\tilde{u} &= \frac{1}{a_{12} + a_{34}} \left[ a_{12}(1 + a_{34}\theta)\bar{w}_1 \right. \\
&\quad \left. + a_{34}(1 - a_{12}\theta)\bar{w}_3 \right] \\
\tilde{v} &= \frac{1}{a_{12} + a_{34}} \left[ a_{12}(1 - a_{34}\theta)w_2 \right. \\
&\quad \left. + a_{34}(1 + a_{12}\theta)w_4 \right].
\end{aligned}$$

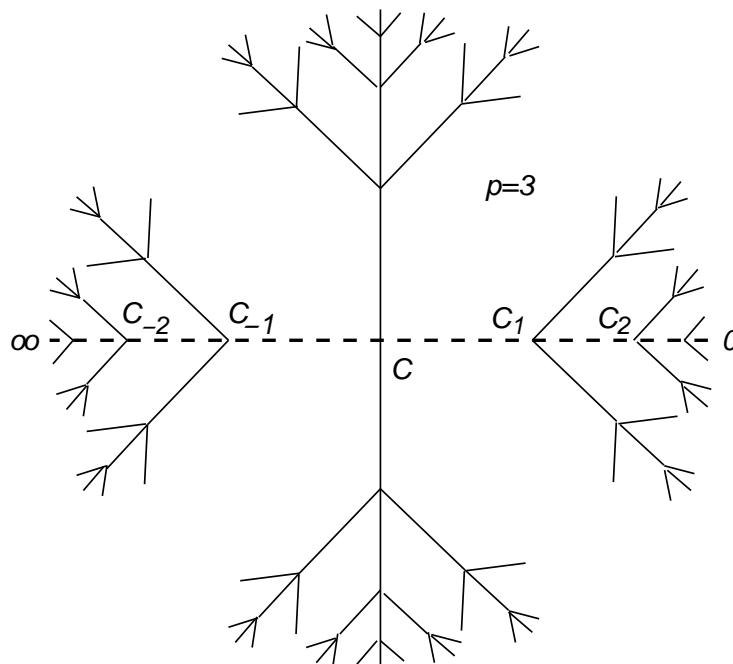
So, not only do the **width** and the **amplitude change**, the **centres** are also **shifted**.



## Noncommutativity from $B$ -field?

**Question:** Is there a microscopic (worldsheet) understanding of the noncommutativity?

Recall that the **worldsheet of the  $p$ -adic string** is an **infinite tree**, a **graph without any loop**. In other words, it is a **Bethe lattice** with  $(p + 1)$  nearest neighbours. [Zabrodin *et al.*]



Worksheet of the 3-adic string:

The tree  $\mathcal{T}_p$  for  $p = 3$ .

The boundary of the tree  $\mathcal{T}_p$  is  $\mathbf{Q}_p$  (through the power series expansion

$$\xi = p^N (\xi_0 + \xi_1 p + \xi_2 p^2 + \dots),$$

$$\xi_n \in \{0, 1, \dots, p-1\}, \xi_0 \neq 0.)$$

The usual string worldsheet is the strip which is *conformally equivalent* to the  $\text{UHP} = \text{SL}(2, \mathbf{R})/\text{SO}(2, \mathbf{R})$ .

$\text{SL}(2, \mathbf{R})$  acts on the boundary of the worldsheet as a symmetry, (and  $\text{SO}(2)$  is its maximal compact subgroup).

The symmetry group of the boundary of the  $p$ -adic worldsheet is  $\text{PGL}(2, \mathbf{Q}_p)$ .

$$\text{PGL}(2, \mathbf{Q}_p)/\text{PGL}(2, \mathbf{Z}_p) = \mathcal{T}_p$$

The ‘worldsheet’ of the  $p$ -adic string is *discrete*, but its boundary is a *continuum*.

$p$ -adic string provides an *exotic discretisation* of the open string worldsheet.

Polyakov action is the discrete lattice action:

$$S_p(X) = \frac{1}{2} \beta_p \sum_{\mathbf{e}} (\delta_{\mathbf{e}} X^\mu) (\delta_{\mathbf{e}} X^\nu) \eta_{\mu\nu}$$

The eqn of motion is the discrete Laplace eqn. Both *Neumann* and *Dirichlet* boundary conditions lead to well defined boundary value problem. Hence, one can have D-branes.

Integrate over the degrees of freedom in the bulk of  $\mathcal{T}_p$  to get a non-local action on the boundary [Spokoiny, Zhang, Parisi]:

$$\mathcal{S}_p = \frac{p(p-1)\beta_p}{4(p+1)} \int_{\mathcal{Q}_p} d\xi d\xi' \frac{(X^\mu(\xi) - X^\mu(\xi'))^2}{|\xi - \xi'|_p^2}.$$

This form will be more useful to us, since a constant *B*-field couple to the boundary. For the usual string, after an integration by parts:

$$\int_{\partial\Sigma} d\xi B_{\mu\nu} X^\mu(\xi) \partial_t X^\nu(\xi)$$

Formally same as the insertion of a background gauge field  $A_\mu [X(\xi)] \sim B_{\mu\nu} X^\nu(\xi)$ .

Now the problem is with a **tangential derivative**. Get around this through the Cauchy-Riemann relation for harmonic/ analytic functions. Define:

$$\partial_t^{(p)} X^\mu(\xi) \equiv \partial_\xi X^\mu = \int_{\mathbf{Q}_p} d\xi' \frac{\text{sgn}_\tau(\xi - \xi')}{|\xi - \xi'|_p^2} X^\mu(\xi')$$

`\begin{digression}`

The field  $\mathbf{Q}_p$  (like  $\mathbf{R}$ ) is *not algebraically closed*. Not all  $\xi \in \mathbf{Q}_p$  has a square-root in it.

Recall, quadratic extension of  $\mathbf{R}$  is  $\mathbf{R}(\sqrt{-1}) = \mathbf{C}$ .

Three inequivalent quadratic extension of  $\mathbf{Q}_p$ :  $\mathbf{Q}_p(\sqrt{\tau})$  for  $\tau = \varepsilon, p$  and  $\tau = \varepsilon p$  ( $p \neq 2$ ), where  $\varepsilon$  is a  $(p - 1)$ -th root of unity.

Define a (real valued) function on  $\mathbf{Q}_p$  (*not* on its extension):

$$\text{sgn}_\tau(\xi) = \begin{cases} +1, & \text{if } \xi = \zeta_1^2 - \tau\zeta_2^2 \\ & \text{for some } \zeta_1, \zeta_2 \in \mathbf{Q}_p, \\ -1 & \text{otherwise} \end{cases}$$

Moreover, if we demand **antisymmetry**

$$\text{sgn}_\tau(-\xi) = -\text{sgn}_\tau(\xi)$$

we must restrict to

$$\tau = p, \varepsilon p$$

$$p = 3 \pmod{4}$$

Generalised gamma-function:

$$\begin{aligned} \Gamma_\tau(s) &= \int_{\mathbf{Q}_p} d\xi e^{2\pi i \xi} |\xi|_p^{s-1} \text{sgn}_\tau(\xi) \\ &= \pm \sqrt{\text{sgn}_\tau(-1)} p^{s-\frac{1}{2}} \end{aligned}$$

`\end{digression}`

Proposed action with  $B$ -field

$$\int_{\mathbb{Q}_p} \frac{\eta_{\mu\nu} (X^\mu(\xi) - X^\mu(\xi')) (X^\nu(\xi) - X^\nu(\xi'))}{|\xi - \xi'|_p^2} d\xi d\xi'$$

$$+ \frac{i(p+1)}{p^2 \Gamma_\tau(-1)} \int_{\mathbb{Q}_p} B_{\mu\nu} X^\mu(\xi) \frac{\text{sgn}_\tau(\xi - \xi')}{|\xi - \xi'|_p^2} X^\nu(\xi') d\xi d\xi'$$

Easy to solve for the  $\text{Green's function}$  in the presence of the  $B$ -field:

$$\mathcal{G}^{\mu\nu}(\xi - \xi') = -G^{\mu\nu} \ln |\xi - \xi'|_p + \frac{i}{2} \theta^{\mu\nu} \text{sgn}_\tau(\xi - \xi'),$$

$$\left( \frac{1}{\eta - iB} \right) = G^{-1} + \frac{i}{2} \frac{p-1}{\alpha' p \ln p \Gamma_\tau(0)} \theta$$

Notice the similarity with the usual bosonic string [[Fradkin-Tseytlin, Abouelsaood et al](#)].

$$\left\langle \prod_{I=1}^N e^{ik^I \cdot X(\xi_I)} \right\rangle_B = \prod_{I < J} \frac{\exp\left(-\frac{i}{2} k^I \theta k^J \text{sgn}_\tau(\xi_I - \xi_J)\right)}{|\xi_I - \xi_J|_p^{-k^I G k^J}}$$

[[Grange, hep-th/0409305](#)]

## Formal consequence of the $B$ -field

This may seem to be the end of the road, since:

$$\begin{aligned} & [X^\mu(0), X^\nu(0)] \\ &= T(\langle X^\mu(0)X^\nu(0_-) \rangle - \langle X^\mu(0)X^\nu(0_+) \rangle) \\ &= \lim_{\substack{\text{sgn}_\tau(\xi)=-1 \\ |\xi|_p \rightarrow 0}} \langle X^\mu(0)X^\nu(\xi) \rangle \\ &\quad - \lim_{\substack{\text{sgn}_\tau(\xi)=+1 \\ |\xi|_p \rightarrow 0}} \langle X^\mu(0)X^\nu(\xi) \rangle \\ &= i\theta^{\mu\nu} \end{aligned}$$

but there are caveats.

There is no natural notion of an order in  $\mathbb{Q}_p$ .

We can define an order by, say, ordering the  $\xi_n$  in  $\xi = p^N \sum \xi_n p^n$ . We can also distinguish between positive and negative  $p$ -adic numbers.

But, this notion is not  $GL(2, \mathbb{Q}_p)$  covariant.

## Tachyon scattering amplitudes

To establish the relation between  $B$ -field and spacetime noncommutativity, one needs the tachyon scattering amplitudes.

### Three-tachyon amplitude

$$\mathcal{A}^{(3)} = \cos \frac{1}{2} k^1 \theta k^2 \equiv c_{12}$$

- There is no projective invariance. But we fix three positions by hand.
- Added and averaged over another term with  $1 \leftrightarrow 2$ .

This agrees with the result from noncommutative deformation of the tachyon effective field theory.



## Four-tachyon amplitude

From  $p$ -adic string theory with  $B$ -field:

$$\begin{aligned} \mathcal{A}^{(4)} = & \frac{p-1}{p} \left( \frac{c_{12}c_{34}}{p^{\alpha(s)}-1} + \frac{c_{13}c_{24}}{p^{\alpha(t)}-1} + \frac{c_{14}c_{23}}{p^{\alpha(u)}-1} \right) \\ & - \frac{1}{p} (c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23}) \\ & + \frac{p+1}{p} c_{12}c_{13}c_{23} \end{aligned}$$

Compare with the result from field theory:

$$\begin{aligned} \mathcal{A}^{(4)} = & \frac{p-1}{p} \left( \frac{c_{12}c_{34}}{p^{\alpha(s)}-1} + \frac{c_{13}c_{24}}{p^{\alpha(t)}-1} + \frac{c_{14}c_{23}}{p^{\alpha(u)}-1} \right) \\ & + \frac{p-2}{p} (c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23}) \end{aligned}$$

Not quite the same, unfortunately!

## Summary

Studied a **noncommutative deformation** of the **exact effective action** of the  $p$ -tachyon.

Obtained a **family of gaussian lump** solution for **all** values of the noncommutativity parameter  $\theta$ . **Smoothly interpolates** between the **soliton** of (commutative)  $p$ -adic string theory and the **noncommutative soliton** of GMS.

These solitons owe their existence to the infinite derivatives in the equation of motion.

Meaningful  $p \rightarrow 1$  limit gives **smoothly interpolating solitons** in **BSFT** of usual bosonic string theory.

Coupled  $B$ -field to the nonlocal action on the **boundary** of the 'worldsheet' and examined its effect on the **spacetime theory**. But ...

## Some remarks

- Our lump solutions for all  $\theta$  is unlike other exact solutions. *E.g.*, Polychronakos, Harvey *et al* obtained a family of solution in a field theory of scalars and gauge fields. But it has no smooth commutative limit.
- The similarity between the solitons of  $p$ -string theory and noncommutative field theories have been noticed before. Dragovich & Volovich noticed that the GMS soliton for  $\theta = 4 \ln 2$  is identical to solitonic 2-brane of the 2-adic string theory. This seems to be a coincidence.

- Still lacking a **worldsheet** understanding of the **noncommutative deformation** in the **space-time effective action**.

We coupled the  **$B$ -field** to the boundary. It can be coupled to the bulk as

$$\int X_*(B) = \text{pull-back of the 2-form } B$$

Problem is that there is no **2-cycle** in the **tree**. However, ...

- Use of noncommutative deformation: tension of D-brane?
- $p$ -adic string in **other backgrounds**?  
Closed  $p$ -strings?

**THANK YOU!**