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Extradimensions and Ultrametricity

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Experimental data belong $\mathbb{Q}$ (rational numbers).

Theoretical models use real $\mathbb{R}$ and complex $\mathbb{C}$ numbers.

$$\mathbb{R} \ni x = \sum_{k=-\infty}^{k_0} x_k 10^k, \quad x_k = 0, 1, .., 9, \quad |x| = \begin{cases} x, & x > 0, \\ -x, & |x| < 0. \end{cases}$$

Uncertainty in distance measuring - quantum gravity:

$$\Delta x \geq l_{pl} = \sqrt{\hbar G / c^3} \approx 10^{-35} m.$$

The hypothesis on non-archimedean nature of space-time at very short distances [I.V.Volovich, preprint CERN-TH.4781/87].

p**-Adic numbers (Kurt Hensel, 1897), (**p **- prime number)**

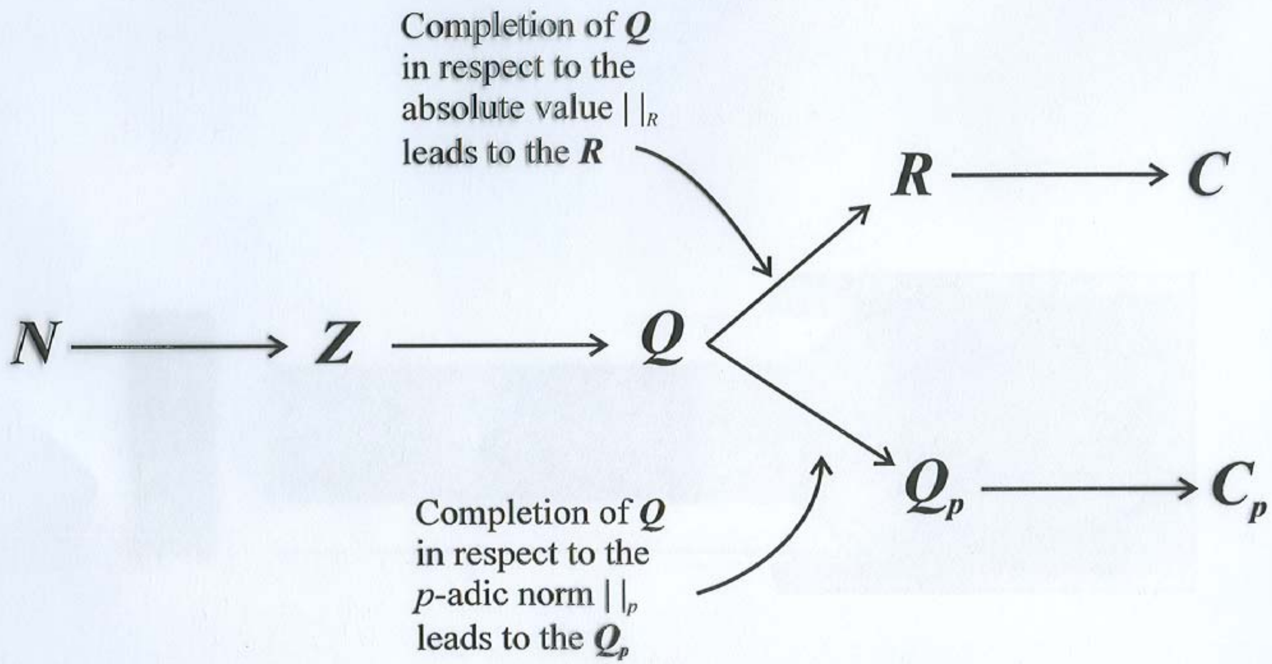
$$\mathbb{Q}_p \ni x = \sum_{k=k_0}^{\infty} x_k p^k, \quad \text{or} \quad x = p^{k_0} \sum_{k=0}^{\infty} x_k p^k \quad x_k = 0, 1, .., p-1.$$

p**-Adic norm:** $|x|_p = p^{-k_0}$, $|x + y|_p \leq max(|x|_p, |y|_p)$.

The p**-adic metric spaces** $(\mathbb{Q}_p, d_p)$, $(d_p = |x - y|_p)$, **as well as real** (R, d), **are: locally compact, complete and separable.**

Differences, $(\mathbb{Q}_p, d_p)$ **are:**

- **non-archimedean (ultrametric)** $d_p(x, z) \leq max(d_p(x, y), d_p(y, z))$,
- **totally disconected ...**



$\boldsymbol{N}$ - natural numbers
$\boldsymbol{Z}$ - integer numbers
$\boldsymbol{Q}$ - rational numbers
$\boldsymbol{R}$ - real numbers

$\boldsymbol{Q}_p$ - p-adic numbers
$\boldsymbol{C}$ - field of complex numbers
(algebraic extension of $\boldsymbol{R}$)
$\boldsymbol{C}_p$ - algebraic extension of $\boldsymbol{Q}_p$

(Ultra)Metric Spaces

- Studying a metric space, it is valuable to have a mental picture that displays distance acurately.

- **Definition**: Let K be a (number) field. A norm valuation on K is a map $|\,|: K \mapsto \mathbb{R}$

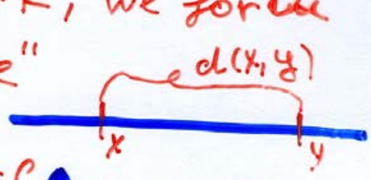
(i) $|x| \geq 0$, $|x| = 0$ iff $x = 0$
(ii) $|x \cdot y| = |x| \cdot |y|$
(iii) $|x + y| \leq |x| + |y|$

An example: standard norm (absolute value)

- The pair $(K, |\,|)$ is a normed (valued) field.

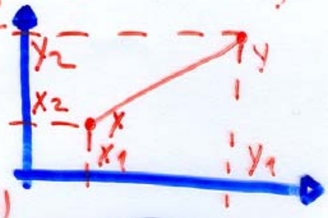
- The map $d(x,y) = |x-y|$ is a metric on K, induced by the norm. (K, d) - metric space.

- When the space is $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, we form a picture on the "number line"



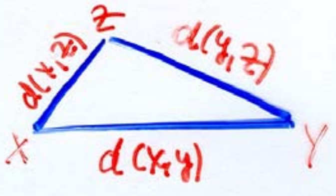
- When the space is $\mathbb{Z}^2$, $\mathbb{Q}^2$, $\mathbb{R}^2$ or $\mathbb{C}$ we use a planar picture

$$d(x,y) = \sqrt{(y_2 - x_2)^2 + (y_1 - x_1)^2}$$



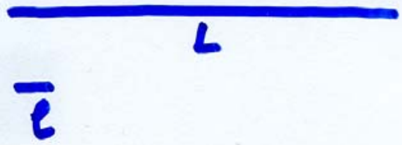
- Nonempty discs (induced topology)
  - closed discs $\{x \in X : d(x,b) \leq r\}$
  - open discs $\{x \in X : d(x,b) < r\}$

- triangle inequality is demonstrated by drawings of triangles



- Archimedean axiom and spaces

$\forall \varepsilon, L \in K,\ \exists n \in N : |n\varepsilon| \geq L$



Ultrametric norm

(iii') $|x+y|_K \leq \max(|x|_K, |y|_K)$, $x, y \in K$

Strong triangle inequality

Ultrametric space (K, d_K), $d_K = |x-y|_K$

$$d_K(x,y) \leq \max\{d_K(x,z), d(y,z)\}$$

- If the space is non-Archimedean (ultrametric) one, then the usual pictures lose their utility.

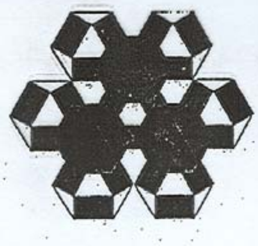
- In such a space:

1) Triangles are always isoceles, with the unequal side (if any) being shortest!

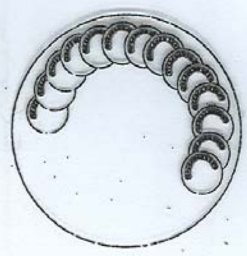
2) Every point in a given disc is a center (of that disc)

3) Two discs can intersect only by having one completely contained in the other!

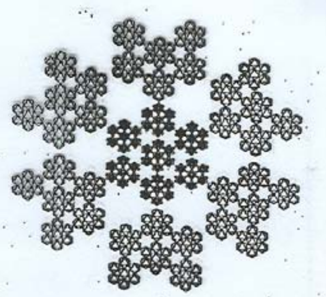
How to imagine and picture to oneself such a space?



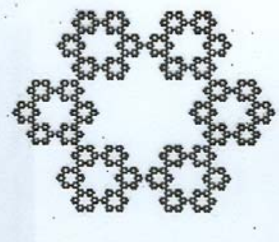
Model Z_{13}



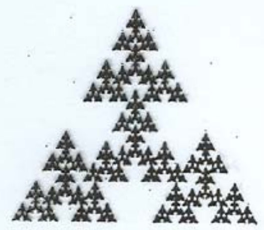
Antoanov model Z_p
($p>30$)



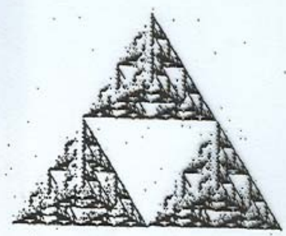
Model dela Q_7



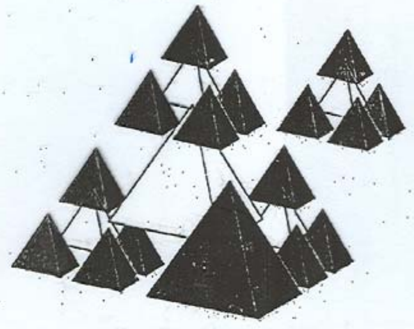
van Kohov model Z_7



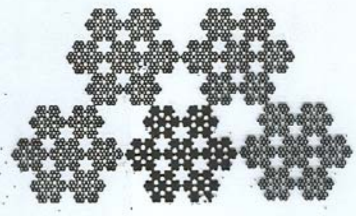
Projekcija modela
($q=2,3$) Z_5



Pogled odozgo na
prostorni model Z_5



Prostorni model Z_5



Model dela Q_7

Adeles (Chevalie, 1930) unify $\mathbb{R}$ and all $\mathbb{Q}_p$

$$a = (a_\infty, a_2, ..., a_p, ...), \quad a_\infty \in \mathbb{R} \equiv \mathbb{Q}_\infty, \quad a_p \in \mathbb{Q}_p.$$

Restriction! For almost all p, $|a_p|_p \leq 1 \Leftrightarrow a_p \in \mathbb{Z}_p$

$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x_p|_p \leq 1\}$ **- p-adic integers.**

An additive character on adelic ring $\mathcal{A}$

$$\chi(xy) = \prod_v \chi_v(x_v y_v) = \exp(-2\pi i x_\infty y_\infty) \prod_p \exp 2\pi i \{x_p y_p\}_p, \;\; x, y \in \mathcal{A}.$$

p-Adic Gauss integral (for all p)

$$\int_{\mathbb{Z}_p} \chi_p(\alpha x^2 + \beta x) dx = \begin{cases} \Omega(|\beta|_p), & |\alpha|_p \leq 1 \\ \lambda_p(\alpha)|2\alpha|_p^{-\frac{1}{2}} \chi_p\left(-\frac{\beta^2}{4\alpha}\right) \Omega\left(|\frac{\beta}{2\alpha}|_p\right), & |4\alpha|_p > 1. \end{cases}$$

Arithmetic function $\lambda_p(a)$ has the following properties

$$\lambda_p(a)\lambda_p(-a) = 1, \;\; \lambda_p(a^2 x) = \lambda_p(x), \;\; \lambda_p(a) = 1 \;\; \text{if} \;\; |a|_p = p^{2k}.$$

Characteristic function

$$\Omega(|x_p|_p) = \begin{cases} 1, & |x_p|_p \leq 1 \Leftrightarrow \quad x_p \in \mathbb{Z}_p, \\ 0, & |x_p|_p > 1 \Leftrightarrow \quad x_p \in \mathbb{Q}_p \backslash \mathbb{Z}_p. \end{cases}$$

Why p-adic and adelic quantum theory?

To investigate space-time on Planck scale (strings, early universe),

To formulate a "fundamental theory" independent on the parametrization of the number field.

ADELIC QUANTUM COSMOLOGY

- The main task of QC is to describe the very early stage in the evolution of the Universe.
- At this stage, the Universe was in a quantum state, which should be described by a wave function (complex valued and depends on some real parameters).
- But, QC is related to Planck scale phenomena - it is natural to reconsider its foundations.
- We maintain here the standard point of view that the wave function takes complex values, but we treat its arguments in a more complete way!
- We regard space-time coordinates, gravitational and matter fields to be adelic, i.e. they have real as well as p-adic properties simultaneously.
- There is no Schroedinger and Wheeler-De Witt equation for cosmological models.
- Feynman's path integral method was exploited [Drag. Nes.] and minisuperspace cosmological models are investigated as a model of adelic quantum mechanics [Drag. Djordj. Nes. Vol.].
- Adelic minisuperspace quantum cosmology is an application of adelic quantum mechanics to the cosmological models.
- Path integral approach to standard quantum cosmology, the starting point is Feynman's path integral method

$$< h''_{ij}, \phi'', \Sigma'' | h'_{ij}, \phi', \Sigma' >_\infty = \int D(g_{\mu\nu})_\infty D(\phi)_\infty \chi_\infty(-S_\infty[g_{\mu\nu}, \phi])$$

The standard 3+1 decomposition for S

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(N^2 - N_i N^i) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j$$

p-adic complex valued cosmological amplitude

$$< h''_{ij}, \phi'', \Sigma'' | h'_{ij}, \phi', \Sigma' >_p = \int D(g_{\mu\nu})_p D(\phi)_p \chi_p(-S_p[g_{\mu\nu}, \phi])$$

p-Adic and Adelic Quantum Mechanics

Standard quantization procedure on $\mathbb{R}$

$$\hat{q}\Psi(x) = x\Psi(x), \qquad x \in \mathbb{R}, \quad \Psi(x) \in \mathbb{C},$$

$$\hat{k}\Psi(x) = -\frac{ih}{2\pi}\frac{d\Psi(x)}{dx}, \qquad [\hat{q}, \hat{k}] = \frac{ih}{2\pi}.$$

$$U(t) = exp\big(- (\frac{2\pi i}{h}) \hat{H} t\big).$$

impossible to generalize on p-adic case $x \in \mathbb{Q}_p, \Psi(x) \in \mathbb{C}$.

Weyl quantization!

$$U_q\Psi(x) = \Psi(x+q),$$

$$V_k\Psi(x) = \chi_p(kx)\Psi(x),$$

$$W(z)\Psi(x) = \chi_p\big(- \frac{kq}{2}\big) U_q V_k \Psi(x) = \chi_p\big(kx + \frac{kq}{2}\big)\Psi(x+q).$$

$$U(t)\Psi(x) = \int \mathcal{K}_t(x,y)\Psi(y)dy.$$

Adelic Quantum Mechanics, (AQM) unifies ordinary and p-adic quantum mechanics [Dragovich, Int. J. Mod. Phys., A10, 2349 (1995)], is defined as a triple

$$(L_2(\mathcal{A}), W(z), U(t))$$

The adelic wave function

$$\Psi(x) = \prod_v \Psi^{(v)}(x_v), \quad x \in \mathcal{A}, \quad (v = \infty, 2, 3, ...)$$

$\Psi^{(p)}(x_p) = \Omega\left(|x_p|_p\right)$ **for all but a finite number of p.**

The evolution operator $U(t)$ and its integral kernel

$$U(t_2,t_1)\Psi(x) = \prod_v U_v(t_2,t_1)\Psi^{(v)}(x) = \prod_v \int_{\mathbb{Q}_v} \mathcal{K}_v(x,t_2;y,t_1)\Psi^{(v)}(y)dy.$$

(4+D)-DIMENSIONAL COSMOLOGICAL MODELS OVER THE FIELD OF REAL NUMBERS

4+*D* – dimensional Kaluza-Klein cosmology with RW type metric

$$ds^2 = -\widetilde{N}^2 dt^2 + R^2(t)\frac{dr_i dr^i}{\left(1+\frac{kr^2}{4}\right)^2} + a^2(t)\frac{d\rho_a d\rho^a}{\left(1+k'\rho^2\right)^2}$$

This model is describing an accelerating universe with dynamical compactification of extra dimensions.

- The form of the energy-momentum tensor is

$$T_{AB} = diag(-\rho, p, p, p, p_D, p_D, \ldots, p_D),$$

If we want the matter is to be confined to the four-dimensional universe, we set all $p_D = 0$.

Energy-momentum tensor

$$p_\chi = \left(\frac{m}{3} - 1\right)\rho_\chi$$

Dimensionally extended Einstein-Hilbert action

$$S = \int \sqrt{-g}\,\widetilde{R}\,dt\,d^3R\,d^D\rho + S_m = \kappa \int dtL$$

$$L = \frac{1}{2\widetilde{N}} Ra^D \dot{R}^2 + \frac{D(D-1)}{12\widetilde{N}} R^3 a^{D-2}\dot{a}^2 + \frac{D}{2\widetilde{N}} R^2 a^{D-1}\dot{R}\dot{a}$$

$$-\frac{1}{2}k\widetilde{N}Ra^D + \frac{1}{6}\widetilde{N}\rho_\chi R^3 a^D$$

Closed universe ($k = 1$), continuity equation

$$\dot{\rho}_\chi R + 3(p_\chi + \rho_\chi)\dot{R} = 0$$

energy density

$$\rho_\chi(R) = \rho_\chi(R_0)\left(\frac{R_0}{R}\right)^m.$$

If we define $\Lambda \equiv \rho_{\chi}(R)$

$$L = \frac{1}{2\widetilde{N}} R a^D \dot{R}^2 + \frac{D(D-1)}{12\widetilde{N}} R^3 a^{D-2} \dot{a}^2 + \frac{D}{2\widetilde{N}} R^2 a^{D-1} \dot{R}\dot{a}$$
$$-\frac{1}{2}\widetilde{N} R a^D + \frac{1}{6}\widetilde{N}\Lambda R^3 a^D$$

Growth of the scaling factor R, leads to the decrease of the cosmological constant

$$\Lambda(R) = \Lambda(R_0)\left(\frac{R_0}{R}\right)^m .$$

If we take $m = 2$, $\Lambda(R_0)R_0^2 = 3$, $\widetilde{N}(t) = R^3(t)a^D(t)N$,

$$L = \frac{1}{2N}\frac{\dot{R}^2}{R^2} + \frac{D(D-1)}{12N}\frac{\dot{a}^2}{a^2} + \frac{D}{2N}\frac{\dot{R}\dot{a}}{Ra}$$

Classical solutions

$$R(t) = Ae^{\alpha t}, \quad a(t) = Be^{\beta t}$$

For $R(0) = a(0) = l_P$

$$R(t) = l_P e^{\alpha t}, \quad a(t) = l_P e^{\beta t}$$

For $D = 1$

$R(t) = l_P e^{Ht}$, $a(t) = l_P e^{-Ht}$, $H = \dot{R}/R - Huble \quad parameter$ accelerating universe, contracting internal space (with same rates).

For $D > 1$

$$R(t) = l_P e^{Ht}, \quad a(t)_\pm = l_P e^{\frac{2Ht}{D}[-1 \pm \sqrt{1-2/3(1-1/D)}]^{-1}}$$

$$R(t)_\pm = l_P e^{\frac{D\beta t}{2}[-1 \pm \sqrt{1-2/3(1-1/D)}]}, \quad a(t) = l_P e^{\beta t}.$$

Quantum solutions

Wheeler-DeWitt equation

$$H\psi(R,a)=0$$

$X=\ln R$, $Y=\ln a$

$$\left[(D-1)\frac{\partial^2}{\partial X^2}+\frac{6}{D}\frac{\partial^2}{\partial Y^2}-6\frac{\partial}{\partial X}\frac{\partial}{\partial Y}\right]\psi(X,Y)=0$$

$x=X\dfrac{3}{D+3}+Y\dfrac{D}{D+3}$, $y=\dfrac{X-Y}{D+3}$

$$\left\{-3\frac{\partial^2}{\partial x^2}+\frac{D+2}{D}\frac{\partial^2}{\partial y^2}\right\}\psi(x,y)=0$$

with four possible solutions

$$\psi_D^{\pm}(x,y)=A^{\pm}e^{\pm\sqrt{\gamma/3}x\pm\sqrt{\gamma D/(D+2)}y},$$

$$\psi_D^{\pm}(x,y)=B^{\pm}e^{\pm\sqrt{\gamma/3}x\mp\sqrt{\gamma D/(D+2)}y}.$$

(4+D)-DIMENSIONAL MODEL OVER THE FIELD OF P-ADIC NUMBERS

Lagrangian of the model

$$L = \frac{1}{2N}\dot{X}^2 + \frac{D(D-1)}{12N}\dot{Y}^2 + \frac{D}{2N}\dot{X}\dot{Y}.$$

Solutions

$$X = C_1 t + C_2,\ Y(t) = C_3 t + C_4.$$

Classical action

$$\bar{S}(X'',Y'',N;X',Y',0)$$
$$= \frac{1}{2N}(X''-X')^2 + \frac{D(D-1)}{12N}(Y''-Y')^2 + \frac{D}{2N}(X''-X')(Y''-Y')$$

Kernel of p-adic operator of evolution

$$K_p(X'',Y'',N;X',Y',0)$$
$$= \lambda_p\left(\frac{D(D+2)}{48N^2}\right)\left|\frac{D(D+2)}{12N^2}\right|_p^{1/2} \chi_p(-\bar{S}(X'',Y'',N;X',Y',0))$$

change $x = X\frac{3}{D+3} + Y\frac{D}{D+3},\ y = \frac{X-Y}{D+3}$

$$\bar{S}(x'',y'',N;x',y',0)$$
$$= \frac{1}{2N}(1+\frac{D(D+5)}{6})(x''-x')^2 - \frac{1}{2N}D(D+3)(y''-y')^2,$$

$$K_p(x'',y'',N;x',y',0) = \lambda_p\left(\frac{6+D(D+5)}{12N}\right)\lambda_p\left(-\frac{D(D+3)}{2N}\right)$$

$$\times\left|\frac{D(D+3)}{2N^2}\left(1+\frac{D(D+5)}{6}\right)\right|_p^{1/2} \chi_p(-\bar{S}).$$

p-adic ground state wave function in (x,y) minisuperspace

$$\psi_p(x,y) = \Omega(|x|_p)\Omega(|y|_p).$$

Conditions

$$|N|_p \leq |1 + D(D+5)/6|_p$$

$$|N|_p \leq |D(D+3)|_p, \; p \neq 2.$$

p-adic ground state wave function in (X,Y) minisuperspace

$$\psi_p(X,Y) = \Omega\left(|(1-\frac{D}{D+3})X + \frac{D}{D+3}Y|_p\right)\Omega\left(|\frac{X-Y}{D-3}|_p\right).$$

2) $\left(p \equiv 3 (\mathrm{mod}\, 4)\right.$, $\lambda_p{}^2(a) = \pm 1$, $|\ell^2|_p = p^\delta$

$$\mathcal{G}_p(x_j|x_i) = \begin{cases} 0 , & \text{for } \delta \geq 2 \iff |x_j - x_i|_p \geq p \\ |h|_p^{-1}(2 - p^{-1}), & \delta = 0 \\ |h|_p^{-1}(p+1)^{-1}\left(\frac{p-1}{p} + 2p|(x_j - x_i)|_p^2\right), & \delta < 0 \end{cases}$$

3) $\mathcal{G}_2(x_j|x_i) = -\dfrac{1}{|h(x_j - x_i)^2|_p}$, for $|(x_j - x_i)|_2 \leq \frac{1}{2}$

$\mathcal{G}_2(x_j|x_i) = 0$, for $x_j = x_i \wedge |x_j - x_i|_2 \geq L$

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