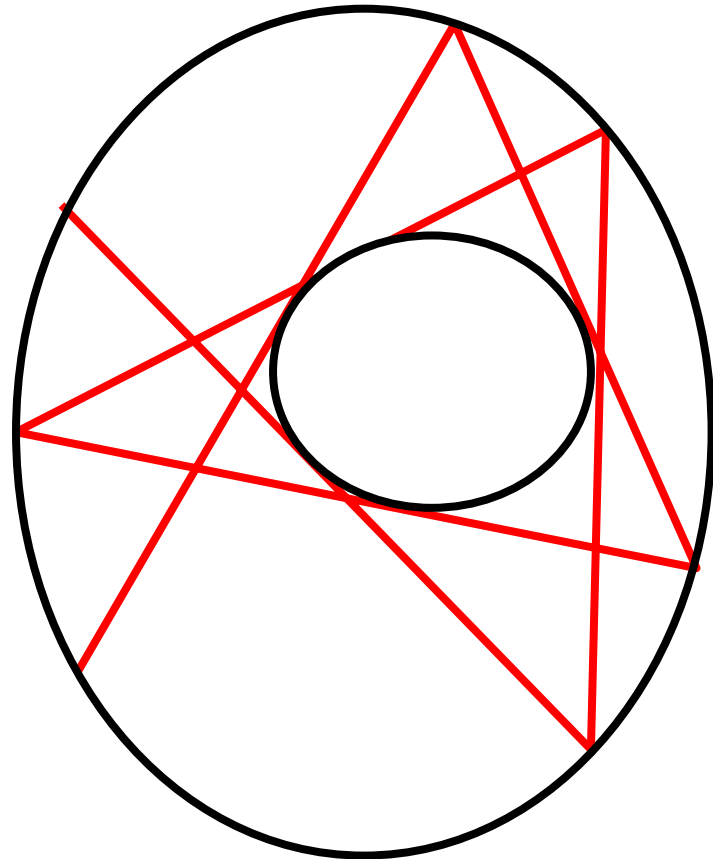


Billiards and pencils of quadrics

Vladimir Dragović, Milena Radnović

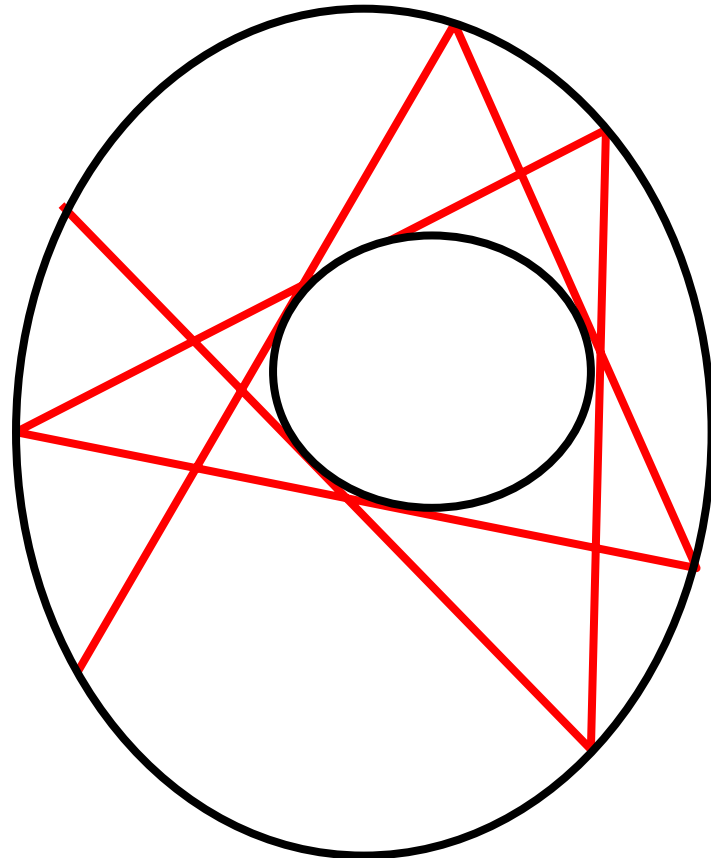
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Poncelet theorem

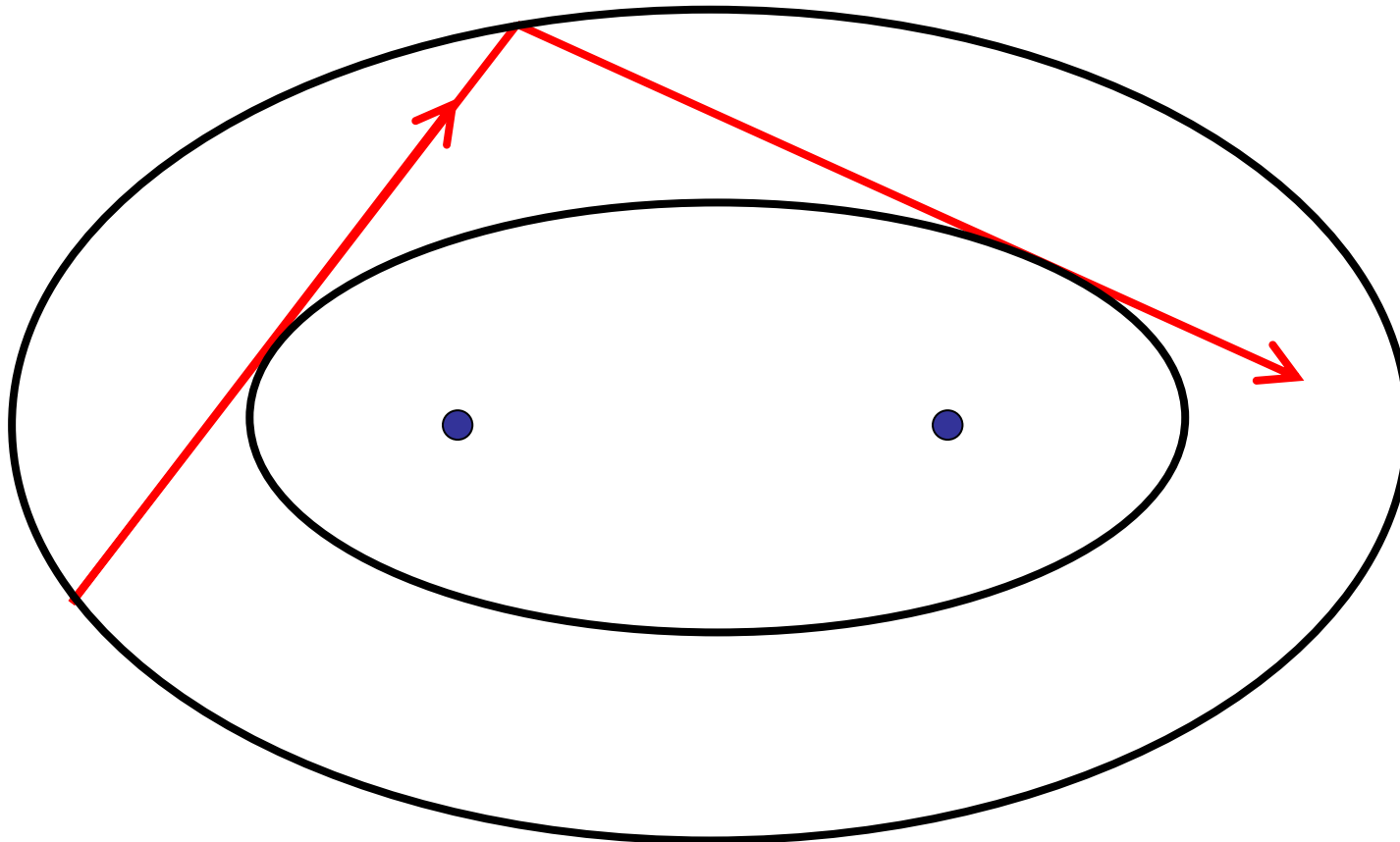


Poncelet theorem

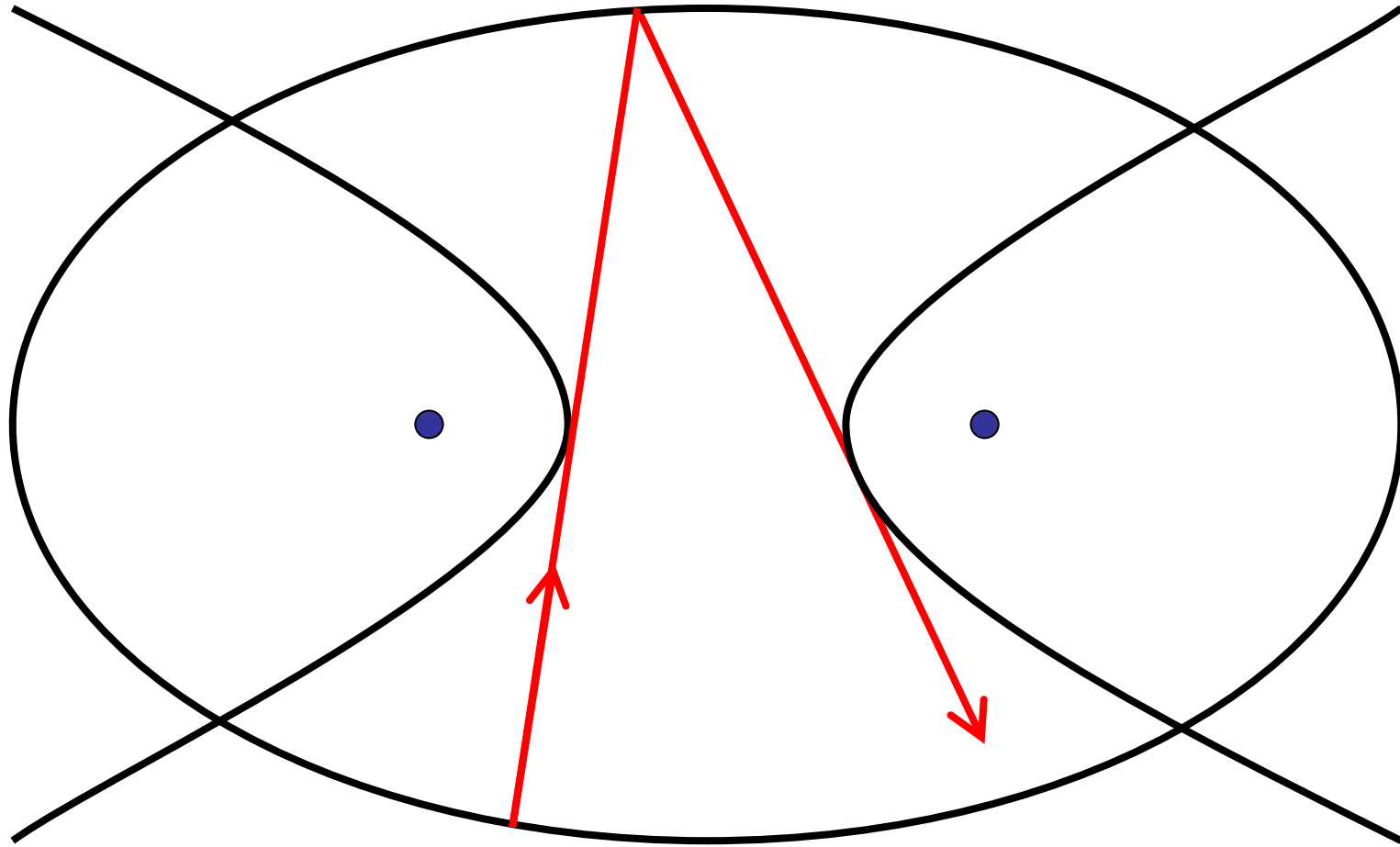
Suppose that two ellipses are given in the plane, together with a closed polygonal line inscribed in one of them and circumscribed about the other one. Then, Poncelet theorem states that infinitely many such closed polygonal lines exist - every point of the first ellipse is a vertex of such a polygon. Besides, all these polygons have the same number of sides.



Mechanical interpretation of Poncelet theorem



Mechanical interpretation of Poncelet theorem



Classical Cayley's condition

Let F be a pencil given by conics C and G .

There exists a polygon with n sides inscribed in G and circumscribed about C if and only if

$$\left| \begin{array}{cccc} C_3 & C_4 & \dots & C_{p+1} \\ C_4 & C_5 & \dots & C_{p+2} \\ & & \dots & \\ C_{p+1} & C_{p+2} & \dots & C_{2p-1} \end{array} \right| = 0$$

for $n=2p$

$$\left| \begin{array}{cccc} C_2 & C_3 & \dots & C_{p+1} \\ C_3 & C_4 & \dots & C_{p+2} \\ & & \dots & \\ C_{p+1} & C_{p+2} & \dots & C_{2p} \end{array} \right| = 0$$

for $n=2p+1$

$$\sqrt{D(x)} = A + Bx + C_2x^2 + C_3x^3 + \dots$$

$D(x)$ is the discriminant of the conic $C+xG$.

Cayley-type condition for billiard inside ellipsoid in arbitrary dimensional space is derived in:

V. Dragović, M. Radnović, *Conditions of Cayley's type for ellipsoidal billiard*, J. Math. Phys. **39** (1998), no. 1, 355-362.

V. Dragović, M. Radnović, *On periodical trajectories of the billiard systems within an ellipsoid in \mathbb{R}^d and generalized Cayley's condition*, J. Math. Phys. **39** (1998), no. 11, 5866-5869.

Recent result:

Cayley-type condition for periodical billiard trajectories inside k confocal quadrics in d -dimensional space.

V. Dragović, M. Radnović, *Cayley-type conditions for billiards within k quadrics in \mathbb{R}^d* , J. of Phys. A: Math. Gen. **37** (2004), 1269-1276.

Billiard inside a domain bounded by confocal quadrics

Theorem *A trajectory of the billiard system within Ω with caustics $Q_{\alpha_1}, \dots, Q_{\alpha_{d-1}}$ is periodic with exactly n_s points at $Q_{\gamma'_s}$ and n_s points at $Q_{\gamma''_s}$ ($1 \leq s \leq d$) if and only if*

$$\sum_{s=1}^d n_s (\mathcal{A}(P_{\gamma'_s}) - \mathcal{A}(P_{\gamma''_s})) = 0$$

on the Jacobian of the curve

$$\Gamma : y^2 = \mathcal{P}(x) := (a_1 - x) \cdots (a_d - x)(\alpha_1 - x) \cdots (\alpha_{d-1} - x).$$

Here, \mathcal{A} denotes the Abel-Jacobi map, where $P_{\gamma'_s}, P_{\gamma''_s}$ are points on Γ with coordinates $P_{\gamma'_s} = (\gamma'_s, (-1)^s \sqrt{\mathcal{P}(\gamma'_s)})$, $P_{\gamma''_s} = (\gamma''_s, (-1)^s \sqrt{\mathcal{P}(\gamma''_s)})$.

Example Consider two domains Ω' and Ω'' in \mathbb{R}^3 . Let Ω' be bounded by the ellipsoid Q_0 and the two-folded hyperboloid Q_β , $a_2 < \beta < a_1$, in such a way that Ω' is placed between the branches of Q_β . On the other hand, suppose Ω'' is bounded by Q_0 , the righthand branch of Q_β (this one which is placed in the half-space $x_1 > 0$) and the plane $x_3 = 0$. Elliptic coordinates of points inside both Ω' and Ω'' satisfy:

$$0 \leq \lambda_3 \leq a_3, \quad \beta \leq \lambda_1 \leq a_1.$$

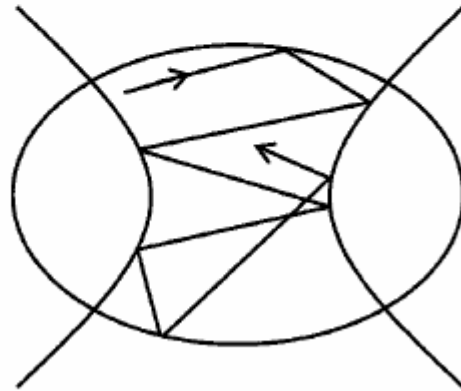
Consider billiard trajectories within these two domains, with caustics Q_{μ_1} and Q_{μ_2} , $a_3 < \mu_1 < a_2$, $a_2 < \mu_2 < a_1$.

In Ω'' , existence of a periodic trajectory with caustics Q_{μ_1} and Q_{μ_2} , which becomes closed after n bounces at Q_0 and $2m$ bounces at Q_β is equivalent to the equality:

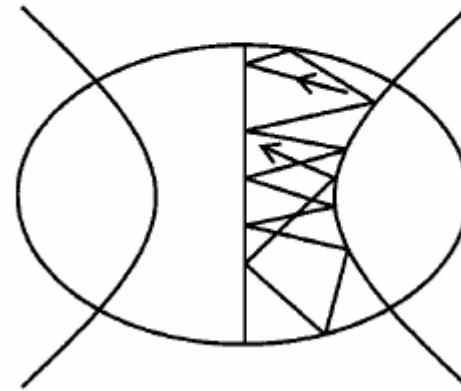
$$n(\mathcal{A}(P_0) - \mathcal{A}(P_{a_3})) + 2m(\mathcal{A}(P_\beta) - \mathcal{A}(P_{\mu_1})) = 0,$$

on the Jacobian of the corresponding hyperelliptic curve. In Ω' , existence of a trajectory with same properties is equivalent to:

$$n(\mathcal{A}(P_0) - \mathcal{A}(P_{a_3})) + 2m(\mathcal{A}(P_\beta) - \mathcal{A}(P_{\mu_1})) = 0 \text{ and } n \text{ is even.}$$



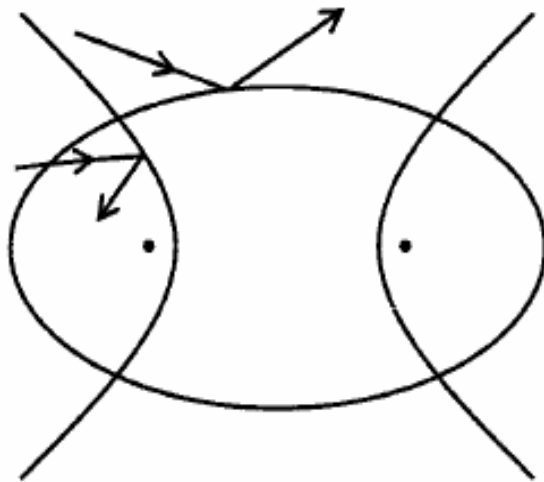
A trajectory inside Ω'



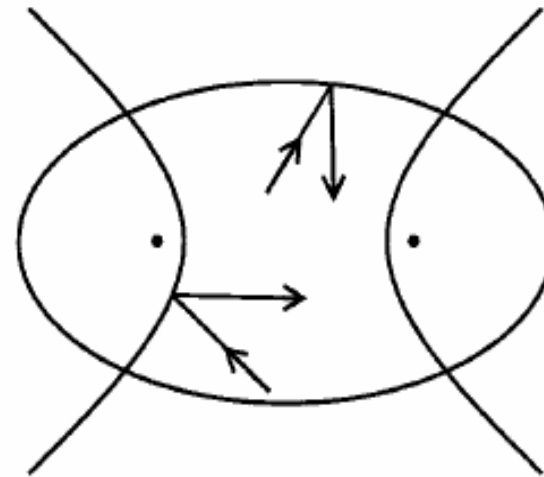
A trajectory inside Ω''

Billiard ordered game

Definition A ray reflects *from outside* at the quadric \mathcal{Q}_λ if the reflection point is a local maximum of the Jacobian coordinate λ_s , and it reflects *from inside* if the reflection point is a local minimum of the coordinate λ_s .



Reflection from outside



Reflection from inside

Definition The billiard ordered game joined to quadrics $Q_{\beta_1}, \dots, Q_{\beta_k}$, with the signature (i_1, \dots, i_k) is the billiard system with trajectories having bounces at $Q_{\beta_1}, \dots, Q_{\beta_k}$ respectively, such that

- the reflection at Q_{β_s} is from inside if $i_s = +1$;
- the reflection at Q_{β_s} is from outside if $i_s = -1$.

Note that any trajectory of a billiard ordered game has $d - 1$ caustics from the same family

Suppose $Q_{\beta_1}, \dots, Q_{\beta_k}$ are ellipsoids and consider a billiard ordered game with the signature (i_1, \dots, i_k) . In order that trajectories of such a game stay bounded, the following condition has to be satisfied:

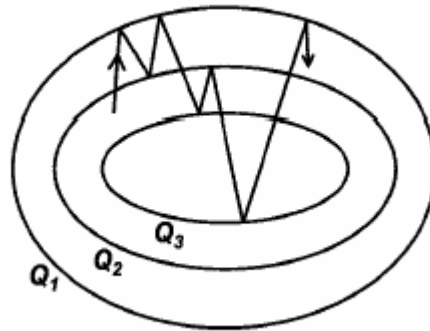
$$i_s = -1 \Rightarrow i_{s+1} = i_{s-1} = 1 \text{ and } \beta_{s+1} < \beta_s, \beta_{s-1} < \beta_s.$$

(Here, we identify indices 0 and $k + 1$ with k and 1 respectively.)

Example On Figure 1 a trajectory corresponding to the 7-tuple

$$(Q_1, Q_2, Q_1, Q_3, Q_2, Q_3, Q_1),$$

with the signature $(1, -1, 1, -1, 1, 1, 1)$, is shown.

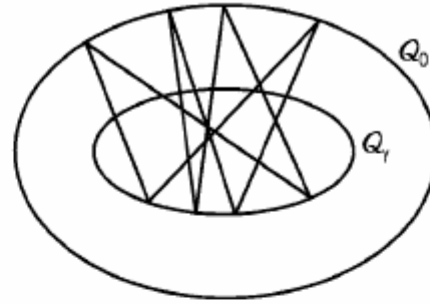


Theorem 5 *Given a billiard ordered game within k ellipsoids $Q_{\beta_1}, \dots, Q_{\beta_k}$ with the signature (i_1, \dots, i_k) . Its trajectory with caustics $Q_{\alpha_1}, \dots, Q_{\alpha_{d-1}}$ is k -periodic if and only if*

$$\sum_{s=1}^k i_s (\mathcal{A}(P_{\beta_s}) - \mathcal{A}(P_{\alpha}))$$

is equal to a sum of several expressions of the form: $(\mathcal{A}(P_{\alpha_p}) - \mathcal{A}(P_{\alpha_{p'}}))$ on the Jacobian of the curve $\Gamma : y^2 = \mathcal{P}(x)$, where $P_{\beta_s} = (\beta_s, +\sqrt{\mathcal{P}(\beta_s)})$, $\alpha = \min\{a_d, \alpha_1, \dots, \alpha_{d-1}\}$ and $Q_{\alpha_p}, Q_{\alpha_{p'}}$ are pairs of caustics of the same type.

When $Q_{\beta_1} = \dots = Q_{\beta_k}$ and $i_1 = \dots = i_k = 1$ we obtain the Cayley-type condition for the billiard motion inside an ellipsoid in \mathbb{R}^d .



A closed trajectory of the billiard ordered game with 8 alternate bounces from inside of two ellipses

The explicit condition for periodicity of such a trajectory is:

$$\text{rank}X < 2,$$

with

$$X_{11} = -4C_0 + C_1\gamma + 3B_1\gamma + 2B_2\gamma^2 + B_3\gamma^3$$

$$X_{12} = -3C_0 + C_1\gamma + 3B_0 + 2B_1\gamma + B_2\gamma^2$$

$$X_{21} = -6C_0 + C_2\gamma^2 + 6B_0 + 6B_1\gamma + 5B_2\gamma^2 + 3B_3\gamma^3$$

$$X_{22} = -6C_0 + C_1\gamma + C_2\gamma^2 + 6B_0 + 5B_1\gamma + 3B_2\gamma^2$$

$$X_{31} = -4C_0 + C_3\gamma^3 + 4B_0 + 4B_1\gamma + 4B_2\gamma^2 + 3B_3\gamma^3$$

$$X_{32} = -4C_0 + C_2\gamma^2 + C_3\gamma^3 + 4B_0 + 4B_1\gamma + 3B_2\gamma^2.$$