Symmetries and integrability for non-autonomous dynamical systems

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2 Point-like type symmetries

- For a p-degrees of freedom dynamical system, an infinitesimal point-like transformation in the $(\vec{q}, t)$ space-time may be defined through an infinitesimal parameter ε :

$$\begin{aligned} t' &= t + \delta t, \;\; \delta t = \varepsilon \xi(\vec{q}, t) \\ q_i' &= q_i + \delta q_i, \;\; \delta q_i = \varepsilon \eta_i(\vec{q}, t) \end{aligned}$$

$$\Downarrow$$

$$\dot{q}_i' \equiv \frac{dq_i'}{dt} = \frac{dq_i + \varepsilon\, d\eta_i}{dt + \varepsilon\, d\xi} = \frac{\dot{q}_i + \varepsilon \dot{\eta}_i}{1 + \varepsilon \dot{\xi}} = \dot{q}_i + \varepsilon(\dot{\eta}_i - \dot{\xi}\dot{q}_i) + O(\varepsilon^2)$$

- Variations of first order in ε:

$$\delta \dot{q}_i = \varepsilon[\dot{\eta}_i(\vec{q}, t) - \dot{\xi}(\vec{q}, t)\dot{q}_i]$$

$$\delta \ddot{q}_i = \varepsilon(\ddot{\eta}_i - 2\dot{\xi}\ddot{q}_i - \ddot{\xi}\dot{q}_i)$$

- For $u(\vec{q}, t)$:

$$\delta u = u(\vec{q}\,', t') - u(\vec{q}, t)$$

$$\delta u = \frac{\partial u}{\partial t}\delta t + \sum_{i=1}^{n} \frac{\partial u}{\partial q_i}\delta q_i = \varepsilon U u(\vec{q}, t),$$

where:

$$U = \xi(\vec{q}, t)\frac{\partial}{\partial t} + \sum_{i=1}^{n} \eta_i(\vec{q}, t)\frac{\partial}{\partial q_i}$$

- For $v(\vec{q}, \dot{\vec{q}}, t)$:

$$\delta v = v(\vec{q}\,', \dot{\vec{q}}\,', t') - v(\vec{q}, \dot{\vec{q}}, t)$$

$$\delta v = \frac{\partial v}{\partial t}\delta t + \sum_{i=1}^{n}(\frac{\partial v}{\partial q_i}\delta q_i + \frac{\partial v}{\partial \dot{q}_i}\delta \dot{q}_i) = \varepsilon U' v(\vec{q}, \dot{\vec{q}}, t)$$

where:

$$U' = U + \sum_{i=1}^{n} \eta_i' \frac{\partial}{\partial \dot{q}_i}, \eta_i' = \dot{\eta}_i - \dot{\xi}\dot{q}_i$$

- The second extension:

$$U" = U' + \sum_{i=1}^{n} \eta_i'' \frac{\partial}{\partial \ddot{q}_i}, \eta_i'' = \ddot{\eta}_i - 2\dot{\xi}\ddot{q}_i - \ddot{\xi}\dot{q}_i$$

3 Lie symmetries for a general non-autonomous dynamical systems

- Case with the eqs.of motion = second order differential eqs.:

$$\ddot{q}_i + \frac{\partial V(\vec{q},t)}{\partial q_i} = 0, i = \overline{1,n}$$

- Lie symmetries ⇒ invariance of the eqs.of motion:

$$U"[\ddot{q}_i + \frac{\partial V(\vec{q},t)}{\partial q_i}] = 0$$

$$\Downarrow$$

$$\eta_i'' + U(\frac{\partial V(\vec{q},t)}{\partial q_i}) = 0$$

$$\Updownarrow$$

$$\ddot{\eta}_i - 2\dot{\xi}\ddot{q}_i - \ddot{\xi}\dot{q}_i + \xi\frac{\partial^2 V}{\partial q_i \partial t} + \sum_{j=1}^{n} \eta_j \frac{\partial^2 V}{\partial q_i \partial q_j} = 0$$

- The previous eqs. imposes:
  - a hierarchy of partial differential equations for $V(\vec{q},t)$
  - expressions for $\xi(\vec{q},t)$ and $\eta_i(\vec{q},t)$ of the form:

$$\xi(\vec{q},t) = \beta(t) + \sum_{i=1}^{n} \alpha_i(t) q_i \quad (1)$$

$$\eta_i(\vec{q},t) = \frac{1}{2}\dot{\beta} q_i + \sum_{j=1}^{n} b_{ij} q_j + \phi_i(t) + q_i \sum_{j=1}^{n} \dot{\alpha}_j q_j \quad (2)$$

where $\beta(t)$, $\phi_i(t), \alpha_i(t)$ must satisfy:

$$\dddot{\beta} q_i + \dot{\beta}[3\frac{\partial V}{\partial q_i} + \sum_j q_j \frac{\partial^2 V}{\partial q_i \partial q_j}] + 2\beta \frac{\partial^2 V}{\partial q_i \partial t} - \quad (3)$$

$$-2\sum_j [b_{ij}\frac{\partial V}{\partial q_j} - \frac{\partial^2 V}{\partial q_i \partial q_j} \sum_k b_{jk} q_k] = 0, \ i,j,k = \overline{1,n}$$

$$\ddot{\phi}_i + \sum_j \phi_j \frac{\partial^2 V}{\partial q_i \partial q_j} = 0, \ i,j = \overline{1,n} \quad (4)$$

$$\sum_j [\ddot{\alpha}_j(2\dot{q}_j q_i + \dot{q}_i q_j) + \alpha_j(2\dot{q}_j \frac{\partial V}{\partial q_i} + \dot{q}_i \frac{\partial V}{\partial q_j})] = 0, \ i,j = \overline{1,n}, \quad (5)$$

$$\sum_j [\dddot{\alpha}_j q_i q_j + \dot{\alpha}_j(q_j \frac{\partial V}{\partial q_i} - q_i \frac{\partial V}{\partial q_j}) + \quad (6)$$

$$+ q_j \frac{\partial^2 V}{\partial q_i \partial q_j} \sum_k \dot{\alpha}_k q_k + \alpha_j \frac{\partial^2 V}{\partial q_i \partial t}] = 0, \ i,j,k = \overline{1,n},$$

- The generator of Lie symmetry takes the expression:

$$U = [\beta(t) + \sum_{i=1}^{n} \alpha_i(t) q_i] \frac{\partial}{\partial t} + \sum_{i=1}^{n} [\frac{1}{2}\dot{\beta} q_i + \sum_{j=1}^{n} b_{ij} q_j + \phi_i(t) + \\ + q_i \sum_{j=1}^{n} \dot{\alpha}_j q_j] \frac{\partial}{\partial q_i} \quad (7)$$

$$\Updownarrow$$

$$U \equiv U_\beta + U_\phi + U_\alpha$$

where

$$U_\phi = \sum_{i=1}^{n} \phi_i(t) \frac{\partial}{\partial q_i} = \sum_{i=1}^{n} U_{\phi_i} \ , \quad (8)$$

$$U_\beta = \beta(t) \frac{\partial}{\partial t} + \sum_{i=1}^{n} [\frac{1}{2}\dot{\beta} q_i + \sum_{j=1}^{n} b_{ij} q_j] \frac{\partial}{\partial q_i} \ , \quad (9)$$

$$U_\alpha = \sum_{i=1}^{n} \alpha_i(t) q_i \frac{\partial}{\partial t} + \sum_{i=1}^{n} [q_i \sum_{j=1}^{n} \dot{\alpha}_j q_j] \frac{\partial}{\partial q_i} \quad (10)$$

- Case with the eqs.of motion = first order differential eqs. $\Longrightarrow U'$ should be used.

- Example:

$$\dot{q}_i + f(q_1, q_2, ...q_n) = 0$$

$$\Downarrow$$

$$U'[\dot{q}_i + f(q_1, q_2, ...q_n)] = 0 \quad (11)$$

This is the case which we will explicitly take into account in the following section.

4 Zonal flows model in anomalous transport

- Aim: study of the anisotropic electrostatic turbulence appearing in turbulent plasmas during the anomalous transport phenomena;
- Method: DCT (the decorrelation trajectory)
- A two-dimensional system, with $\{x,\ y,\ z \equiv k_x,\ u \equiv k_y\}$ ⇒Langevin-type equations:

$$\begin{aligned}
\dot{x} + 2A\frac{zu}{(1+z^2+u^2)^2} + f(t)\frac{\partial \mathcal{E}(x,y)}{\partial y} &= 0 \\
\dot{y} - A\frac{1+z^2-u^2}{(1+z^2+u^2)^2} - f(t)\frac{\partial \mathcal{E}(x,y)}{\partial x} &= 0 \\
\dot{z} - zf(t)\frac{\partial \mathcal{E}^2(x,y)}{\partial x \partial y} + uf(t)\frac{\partial \mathcal{E}^2(x,y)}{\partial x^2} &= 0 \\
\dot{u} - zf(t)\frac{\partial \mathcal{E}^2(x,y)}{\partial y^2} + uf(t)\frac{\partial \mathcal{E}^2(x,y)}{\partial x \partial y} &= 0
\end{aligned}$$

  A is an arbitrary constant and $\mathcal{E}(x,y)$ may be a third order polynomial function.

- In our study we will consider $\mathcal{E}(x,y)$ of the form:

$$\mathcal{E}(x,y) = a(x^3+y^3) + b(x^2y + yx^2),\ \ a, b \text{ arbitrary constants}$$

- For $a = \frac{b}{3}$:

$$\frac{\partial \mathcal{E}^2(x,y)}{\partial x^2} = \frac{\partial \mathcal{E}^2(x,y)}{\partial x \partial y} = \frac{\partial \mathcal{E}^2(x,y)}{\partial y^2} = 2b(x+y)$$

- For the Poisson surface of section $u = 0$:

$$\begin{aligned}
\dot{x} + f(t)b(x+y)^2 &= 0 \\
\dot{y} - \frac{A}{(1+z^2)} - f(t)b(x+y)^2 &= 0 \\
\dot{z} - 2f(t)bz(x+y) &= 0
\end{aligned}$$

4.1. Lie symmetries for the model

- The infinitesimal form of such symmetries may be written as:

$$\begin{aligned} t &\to t + \varepsilon\xi(t,x,y,z) \\ x &\to x + \varepsilon\eta_1(t,x,y,z) \\ y &\to y + \varepsilon\eta_2(t,x,y,z) \\ z &\to z + \varepsilon\rho(t,x,y,z) \end{aligned}$$

- The infinitesimal generator of the Lie symmetries would have the form:

$$U = \xi(\vec{r},t)\frac{\partial}{\partial t} + \eta_1(\vec{r},t)\frac{\partial}{\partial x} + \eta_2(\vec{r},t)\frac{\partial}{\partial y} + \rho(\vec{r},t)\frac{\partial}{\partial z}, \ \vec{r} = (x,y,z)$$

- The first extension:

$$U' = U + (\dot{\eta}_1 - \dot{x}\dot{\xi})\frac{\partial}{\partial \dot{x}} + (\dot{\eta}_2 - \dot{y}\dot{\xi})\frac{\partial}{\partial \dot{y}} + (\dot{\rho} - \dot{z}\dot{\xi})\frac{\partial}{\partial z}$$

- The invariance conditions give:

$$\xi\dot{f}(t)b(x+y)^2 + 2\eta_1 f(t)b(x+y) + 2\eta_2 f(t)b(x+y) + \frac{\partial\eta_1}{\partial t} +$$

$$(\frac{\partial\eta_1}{\partial x} - \frac{\partial\xi}{\partial t})\dot{x} + \frac{\partial\eta_1}{\partial y}\dot{y} + \frac{\partial\eta_1}{\partial z}\dot{z} - \frac{\partial\xi}{\partial x}\dot{x}^2 - \frac{\partial\xi}{\partial y}\dot{x}\dot{y} - \frac{\partial\xi}{\partial z}\dot{x}\dot{z} = 0$$

$$\begin{aligned} \xi &= \xi(t) \\ \eta_1(x,t) &= \dot{\xi}(t)x + B(t), B(t)\text{arbitrary function} \end{aligned}$$

$$\Downarrow$$

$$\xi\dot{f}(t)b(x+y)^2 + 2f(t)b(x+y)[\dot{\xi}(t)x + B(t)] +$$
$$+2\eta_2 f(t)b(x+y) + \ddot{\xi}(t)x + \dot{B}(t) = 0$$

- The second invariance requirement:

$$-\xi\dot{f}(t)b(x+y)^2 - 2f(t)b(x+y)[\dot{\xi}(t)x + B(t)] -$$

$$-2\eta_2 f(t)b(x+y) + 2A\rho\frac{z}{(1+z^2)^2} + \frac{\partial\eta_2}{\partial t} +$$

$$+(\frac{\partial\eta_2}{\partial y} - \frac{d\xi}{dt})\dot{y} + \frac{\partial\eta_2}{\partial x}\dot{x} + \frac{\partial\eta_2}{\partial z}\dot{z} = 0$$

$$\eta_2(t,y) = \dot{\xi}(t)y + C(t), C(t) \text{ arbitrary constant}$$
$$A = 0$$
$$\Downarrow$$
$$-\xi\dot{f}(t)b(x+y)^2 - 2f(t)b(x+y)[\dot{\xi}(t)x + B(t)] -$$

$$-2bf(t)(x+y)[\dot{\xi}(t)y+C(t)]+\ddot{\xi}(t)y+\dot{C}(t)=0$$
$$\xi\dot{f}(t)b(x+y)^2+2f(t)b(x+y)[\dot{\xi}(t)x+B(t)]+$$
$$+2f(t)b(x+y)[\dot{\xi}(t)y+C]+\ddot{\xi}(t)x+\dot{B}(t)=0$$
$$\Downarrow$$
$$\ddot{\xi}(t)[x+y]+\dot{B}(t)+\dot{C}(t)=0$$

$$\begin{aligned}\ddot{\xi}(t) &= 0\\ \dot{B}(t) &= -\dot{C}(t)\end{aligned}$$

$$\Downarrow$$

$$\xi(t)=mt+n,\ m.n \text{ arbitrary constants}$$

- The third invariance condition:

$$-2\xi\dot{f}(t)bz(x+y)-2f(t)bz[\dot{\xi}(t)x+B(t)]-2f(t)bz[\dot{\xi}(t)y+C(t)]-$$
$$-2\rho f(t)b(x+y)+\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial x}\dot{x}+\frac{\partial\rho}{\partial y}\dot{y}+(\frac{\partial\rho}{\partial x}-\frac{d\xi}{dt})\dot{z}=0$$
$$\rho(t,z)=\dot{\xi}(t)z+D(t);D(t)=arbitrary\ function$$
$$-2\xi\dot{f}(t)bz(x+y)-2f(t)bz[\dot{\xi}(t)x+B(t)]-2f(t)bz[\dot{\xi}(t)y+C(t)]-$$
$$-2f(t)b(x+y)[\dot{\xi}(t)z+D(t)]+\ddot{\xi}(t)z+\dot{D}(t)=0$$

$$\Downarrow$$

$$\begin{aligned}\xi\dot{f}(t)+2f(t)\dot{\xi}(t) &= 0\\ D(t) &= 0\end{aligned}$$

$$(mt+n)\dot{f}(t)+2f(t)m=0$$

- Conclusion: the first extension of the generator for Lie symmetries has the expression:

$$\begin{aligned}U' &= (mt+n)\frac{\partial}{\partial t}+[mx+B(t)]\frac{\partial}{\partial x}+[my-B(t)]\frac{\partial}{\partial y}+mz\frac{\partial}{\partial z},\\ m,n,B, &= \text{arbitrary constants}\end{aligned}$$

- Particular values for the arbitrary constants:

$i)\ m=1,n=0,B=0\Rightarrow U_1=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\Rightarrow f(t)=\frac{c}{t^2}$
$ii)\ n=1,m=0,B=0\Rightarrow U_2=\frac{\partial}{\partial t}\Longrightarrow f(t)=cons\tan t$
$iii)\ B=1,n=0,m=0\Rightarrow U_3=\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\Longrightarrow f(t)=arbitrary$

4.2 Lie invariants

- The invariants associated to $U_1' \equiv U$:

$$U_1' I(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) = 0$$

$$\Downarrow$$

$$t\frac{\partial I}{\partial t} + x\frac{\partial I}{\partial x} + y\frac{\partial I}{\partial y} + z\frac{\partial I}{\partial z} = 0$$

- Possible solution:

$$I = \frac{x\dot{x}}{t}g_1(\dot{y}, \dot{z}) + \frac{y\dot{y}}{t}g_2(\dot{x}, \dot{z}) + \frac{z\dot{z}}{t}g_3(\dot{x}, \dot{y})$$

where $g_1(\dot{y}, \dot{z})$, $g_2(\dot{x}, \dot{z})$, $g_3(\dot{x}, \dot{y})$ are arbitrary functions in regard to their arguments

Conclusions:

- ($\ni$) invariant quantity linear in each component of the velocity
- Generally, invariant quantities could be obtained by imposing a priori dependence on the coordinates or velocities.
- Despite the dominance of the chaotic behavior, the system presents three Lie symmetries. For one of these symmetry operators, the existence of a class of invariant quantities, linear in coordinates has been pointed out.

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