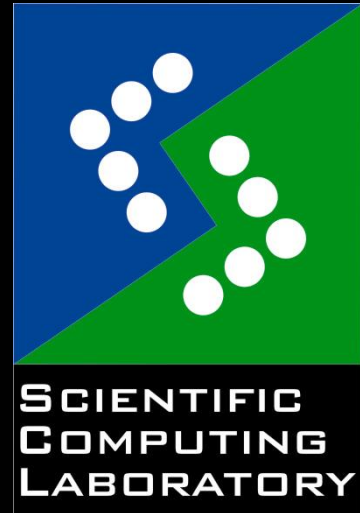


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EULER SUMMATION FORMULA FOR PATH INTEGRALS

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ORDINARY INTEGRALS (1)

■ DEFINITION

$$I[f] \equiv \int_0^T f(t)dt = \lim_{N \rightarrow \infty} I_N[f]$$

$$I_N[f] = \sum_{n=1}^N f(t_n)\epsilon_N$$

$$\epsilon_N = T/N \quad t_n = n\epsilon_N$$

■ BAD FOR BRUTE FORCE NUMERICAL CALCULATIONS

$$I_N[f] = I[f] + O(1/N)$$

ORDINARY INTEGRALS (2)

- TO ANALYTICALLY SOLVE EVEN THE SIMPLEST INTEGRALS YOU NEEDED TO:
 - FIND USEFUL DISCRETIZATION
 - DO GENERAL N-FOLD SUM
 - DO THE CONTINUUM LIMIT
- EULER'S SUMMATION FORMULA
 - SPEEDS UP CONVERGENCE TO THE CONTINUUM LIMIT AS FAST AS YOU WANT
 - WASN'T OF MUCH USE NUMERICALLY – NO COMPUTERS IN THE 18TH CENTURY
 - POINTED TO AN UNDERLYING SIMPLICITY – PRECURSOR TO INTEGRATION THEORY
- RIEMMANN'S INTEGRATION THEORY

EULER'S FORMULA (1)

- DISCRETIZATION IS NOT UNIQUE.
INSTEAD OF $f(t)$ WE CONSTRUCT AN
EQUIVALENT FUNCTION $f^*(t; \epsilon_N)$:

$$f^*(t; \epsilon_N) \rightarrow f(t) \quad (\text{IN CONTINUUM LIMIT})$$

SUCH THAT

$$I_N[f^*] = I[f] \quad (\text{FOR ALL } N)$$

EULER'S FORMULA (2)

- **STEP 1:** $f(t) = 1$

$$I_N[1] = \sum_{n=1}^N \epsilon_N = T$$

THUS $f^* - f$ DEPENDS ONLY ON \dot{f}, \ddot{f}, \dots

- **STEP 2:** $f(t) = t$

$$I_N[t] = \sum_{n=1}^N t_n \epsilon_N = \frac{T^2}{2} + \frac{T^2}{2N}$$

$$I_N\left[t - \frac{\epsilon_N}{2}\right] = \frac{T^2}{2}$$

THUS $f^* - f + \frac{\epsilon_N}{2} \dot{f}$ DEPENDS ONLY ON \ddot{f}, \dots

EULER'S FORMULA (3)

- **STEP 3:** $f(t) = t^2$

$$I_N[t^2] = \sum_{n=1}^N t_n^2 \epsilon_N = \frac{T^3}{3} + \frac{T^3}{2N} + \frac{T^3}{6N^2}$$

$$I_N[t^2 - \epsilon_N t_n - \frac{2}{3} \epsilon_N^2] = \frac{T^3}{3}$$

THUS $f^* = f - \frac{\epsilon_N}{2} \dot{f} - \frac{2\epsilon_N^2}{3} \ddot{f} + \dots$

- **ETC.**

EULER'S FORMULA (4)

- THEREFORE

$$\int_0^T f(t)dt = \sum_{n=1}^N f(t_n)\epsilon_N - \frac{\epsilon_N}{2} \sum_{n=1}^N \dot{f}(t_n)\epsilon_N - \frac{2\epsilon_N^2}{3} \sum_{n=1}^N \ddot{f}(t_n)\epsilon_N + \dots$$

- WE DENOTE THE FIRST p TERMS OF f^* BY $f^{(p)}$. THEN

$$I[f] = I_N[f^{(p)}] + O(1/N^p)$$

PATH INTEGRALS (1)

- **AMPLITUDES ARE THE $N \rightarrow \infty$ LIMIT OF**

$$A_N(a, b; T) = \left(\frac{1}{2\pi\epsilon_N} \right)^{\frac{N}{2}} \int dq_1 \cdots dq_{N-1} e^{-S_N}$$

- **ACTION** $S = \int_0^T dt \left(\frac{1}{2} \dot{q}^2 + V(q) \right)$

- **NAÏVE DISCRETIZED ACTION**

$$S_N = \sum_{n=0}^{N-1} \left(\frac{\delta_n^2}{2\epsilon_N} + \epsilon_N V_n \right)$$

$$\delta_n = q_{n+1} - q_n \quad V_n = V(\bar{q}_n) \quad \bar{q}_n = \frac{1}{2}(q_{n+1} + q_n)$$

PATH INTEGRALS (2)

- **CONSTRUCT EQUIVALENT DISCRETIZED EFFECTIVE ACTION**

$$S_N^* = \sum_{n=0}^{N-1} \left(\frac{\delta_n^2}{2\epsilon_N} + \epsilon_N W_n^* \right)$$

SUCH THAT

$$A_N^*(a, b; T) = A(a, b; T)$$

- **WE FIRST DERIVE SOME GENERAL PROPERTIES OF W^***

PATH INTEGRALS (3)

- ALL AMPLITUDES MAY BE WRITTEN AS

$$A(q_{n+1}, q_n; \epsilon_N) = \left(\frac{1}{2\pi\epsilon_N} \right)^{\frac{1}{2}} \exp \left(-\frac{\delta_n^2}{2\epsilon_N} \right) \mathcal{A}(q_{n+1}, q_n; \epsilon_N)$$

WHERE $\mathcal{A} \rightarrow 1$ AS $\epsilon_N \rightarrow 0$.

- LINEARITY OF QUANTUM STATES GIVES

$$A(a, b; T) = \int dq_1 \cdots dq_{n-1} A(b, q_{n-1}; \epsilon_N) \cdots A(q_1, a; \epsilon_N)$$

- FROM THIS WE FIND

$$\exp(-\epsilon_N W_n^*) = \mathcal{A}(q_{n+1}, q_n; \epsilon_N)$$

PATH INTEGRALS (4)

- ALTHOUGH W_n^* IS REMINISCENT OF AN EFFECTIVE POTENTIAL IT DOES NOT ONLY DEPEND ON \bar{q}_n BUT ON δ_n AND ϵ_N

$$W_n^* = W^*(\delta_n, \bar{q}_n; \epsilon_N)$$

- REALITY OF (EUCLIDEAN) AMPLITUDES GIVES

$$\begin{aligned} A(a, b; T) &= A(a, b; T)^\dagger = \\ &= \langle b | e^{-T\hat{H}} | a \rangle^\dagger = \langle a | e^{-T\hat{H}} | b \rangle = A(b, a; T) \end{aligned}$$

PATH INTEGRALS (5)

- THIS IMPLIES

$$W^*(\delta_n, \bar{q}_n; \epsilon_N) = W^*(-\delta_n, \bar{q}_n; \epsilon_N)$$

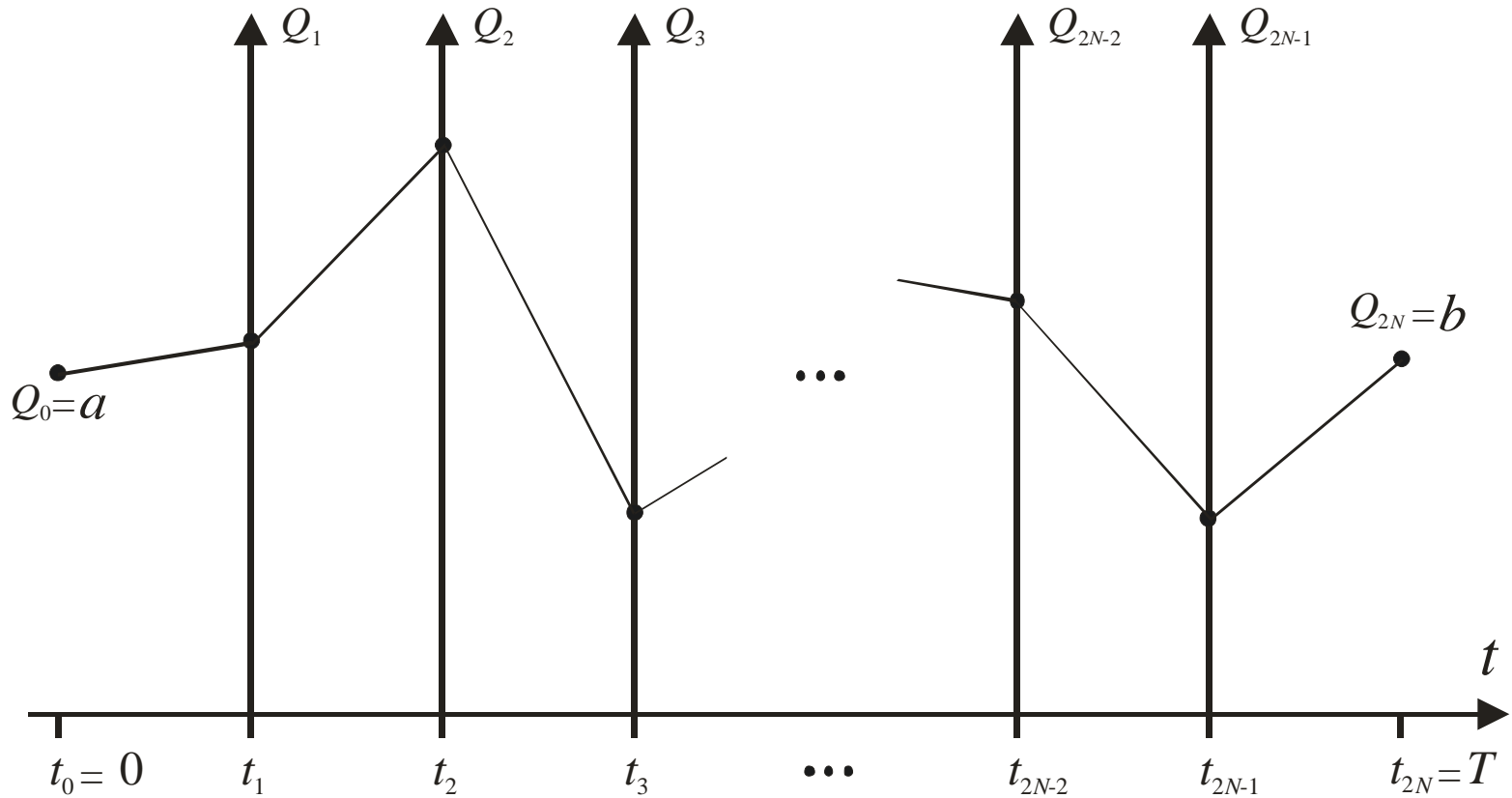
- OR, SAID ANOTHER WAY

$$\begin{aligned} W^*(\delta_n, \bar{q}_n; \epsilon_N) &= \\ &= g_0(\bar{q}_n; \epsilon_N) + \delta_n^2 g_1(\bar{q}_n; \epsilon_N) + \delta_n^4 g_2(\bar{q}_n; \epsilon_N) + \dots \end{aligned}$$

ALL THE g_k ARE REGULAR IN THE CONTINUUM LIMIT; ALSO

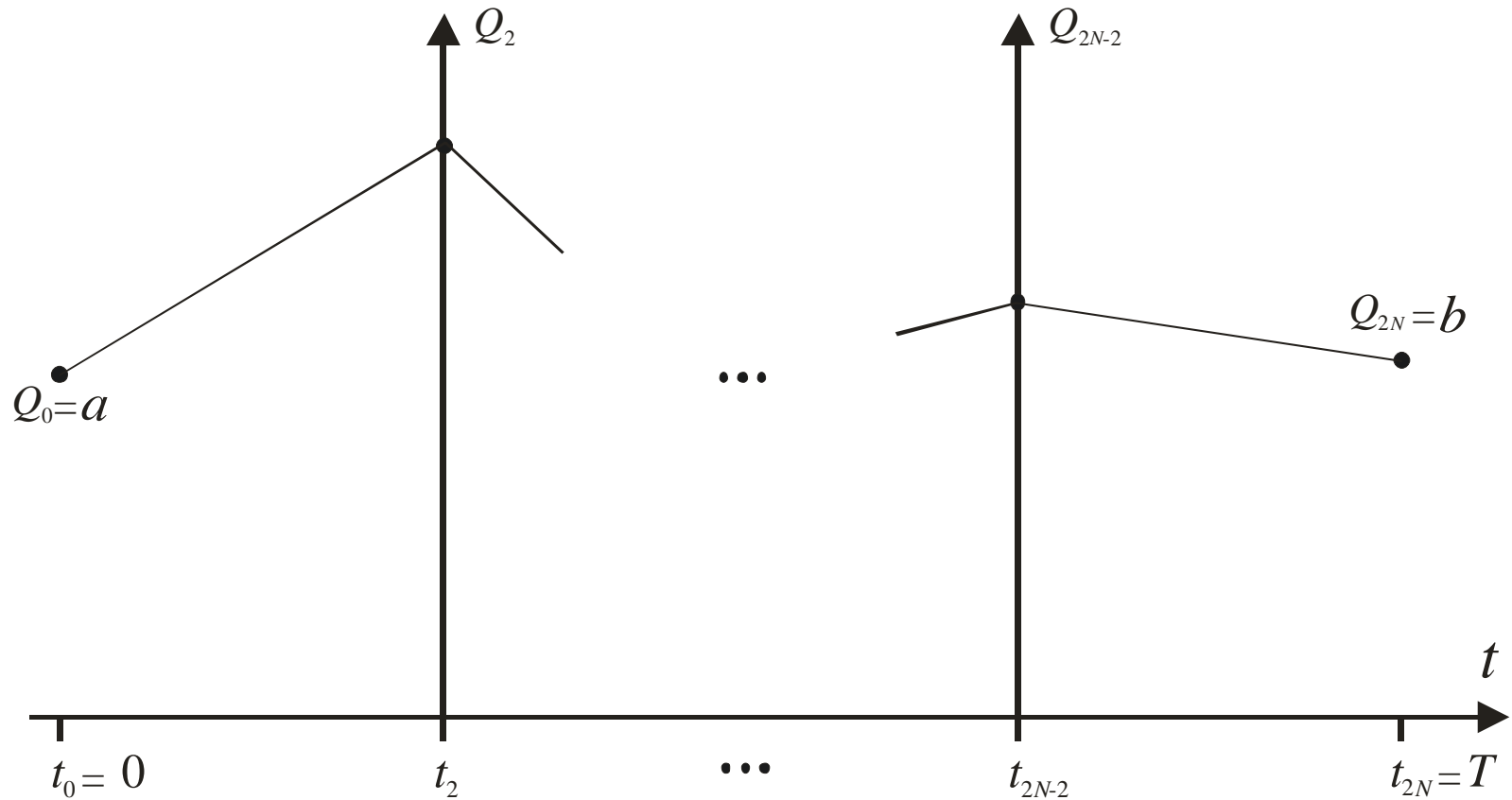
$$g_0(\bar{q}_n; \epsilon_N) \rightarrow V(\bar{q}_n)$$

COMPARE DISCRETIZATIONS (1)



DISCRETIZATION WITH $2N$ TIME STEPS

COMPARE DISCRETIZATIONS (2)



COARSER DISCRETIZATION WITH N TIME STEPS

COMPARE DISCRETIZATIONS (3)

- INTEGRATING OUT EVERY OTHER POINT

$$e^{-\tilde{S}_N} = \left(\frac{2}{\pi \epsilon_N} \right)^{\frac{N}{2}} \int dx_1 \cdots dx_N e^{-S_{2N}}$$

- THE $2N$ -FOLD DISCRETIZED COORDINATES Q_0, Q_1, \dots, Q_{2N} ARE GIVEN IN TERMS OF N -FOLD COORDINATES q_0, q_1, \dots, q_N AND THE INTERMEDIATE POINTS x_1, x_2, \dots, x_N ACCORDING TO $Q_{2k} = q_k$ AND $Q_{2k-1} = x_k$

COMPARE DISCRETIZATIONS (4)

- N -FOLD AMPLITUDES CALCULATED WITH \tilde{S} GIVE THE SAME ANSWER AS $2N$ -FOLD AMPLITUDES CALCULATED WITH S . IF ONLY WE COULD ACTUALLY DO THE INTEGRALS!
- BIGGEST PROBLEM: EVEN IF WE COULD INTEGRATE THEM ONCE WE COULDN'T ITERATE THIS PROCEDURE SINCE \tilde{S} AND S DO NOT HAVE THE SAME FUNCTIONAL FORM.
- SOLUTION: USE S^* (ON BOTH SIDES).

COMPARE DISCRETIZATIONS (5)

- GET INTEGRAL EQUATION

$$\begin{aligned} \exp[-\epsilon_N W^*(\delta_n, \bar{q}_n; \epsilon_N)] &= \\ &= \left(\frac{2}{\pi\epsilon_N}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} dy \exp\left(-\frac{2}{\epsilon_N} y^2\right) F\left(\bar{q}_n + y; \frac{\epsilon_N}{2}\right) \end{aligned}$$

WHERE

$$\begin{aligned} -\frac{2}{\epsilon_N} \ln F(x; \epsilon_N) &= g_0\left(\frac{q_{n+1} + x}{2}; \epsilon_N\right) + g_0\left(\frac{x + q_n}{2}; \epsilon_N\right) + \\ &+ (q_{n+1} - x)^2 g_1\left(\frac{q_{n+1} + x}{2}; \epsilon_N\right) + (x - q_n)^2 g_1\left(\frac{x + q_n}{2}; \epsilon_N\right) + \dots \end{aligned}$$

- WORKS FOR FREE PARTICLE AND HARMONIC OSCILLATOR. IN GENERAL ASYMPTOTICALLY EXPAND - SIMPLER THAN LOOP EXPANSION. SMALL PARAMETER IS NOW ϵ_N

COMPARE DISCRETIZATIONS (6)

- FIND

$$g_0(\bar{q}_n; \epsilon_N) + \delta_n^2 g_1(\bar{q}_n; \epsilon_N) + \delta_n^4 g_2(\bar{q}_n; \epsilon_N) + \dots =$$
$$= -\frac{1}{\epsilon_N} \ln \left[\sum_{m=0}^{\infty} \frac{F^{(2m)}\left(\bar{q}_n; \frac{\epsilon_N}{2}\right)}{(2m)!!} \left(\frac{\epsilon_N}{4}\right)^m \right]$$

- ALL THAT REMAINS IS THE TEDIOUS TASK OF CALCULATING ALL THE DERIVATIVES AND EXPANDING ALL THE g_k AROUND THE MID POINT \bar{q}_n (EASILY DONE WITH A PROGRAM LIKE MATHEMATICA). WE HAVE DONE THIS TO LEVEL $p=9$, I.E. TO $O(1/N^p)$

COMPARE DISCRETIZATIONS (7)

- FOR EXAMPLE, TO LEVEL $p=3$ WE GET

$$g_0(\bar{q}_n; \epsilon_N) = g_0\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) + \epsilon_N \left[\frac{1}{4} g_1\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) + \frac{1}{32} g_0''\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) \right] + \\ + \epsilon_N^2 \left[\frac{3}{16} g_2\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) - \frac{1}{32} g_0'^2\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) + \frac{1}{2048} g_0^{(4)}\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) + \frac{3}{128} g_1''\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) \right]$$

$$g_1(\bar{q}_n; \epsilon_N) = \frac{1}{4} g_1\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) + \frac{1}{32} g_0''\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) + \\ + \epsilon_N \left[\frac{3}{8} g_2\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) + \frac{1}{1024} g_0^{(4)}\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) - \frac{1}{64} g_1''\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) \right]$$

$$g_2(\bar{q}_n; \epsilon_N) = \frac{1}{16} g_2\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) + \frac{1}{6144} g_0^{(4)}\left(\bar{q}_n; \frac{\epsilon_N}{2}\right) + \frac{1}{128} g_1''\left(\bar{q}_n; \frac{\epsilon_N}{2}\right)$$

- THE SYSTEM WAS DERIVED AS AN EXPANSION IN POWERS OF ϵ_N . EXPANDING ALL THE g_k APPROPRIATELY WE EASILY FIND THE SOLUTION AT THE APPROPRIATE LEVEL p .

COMPARE DISCRETIZATIONS (7)

- THE LEVEL $p=3$ SOLUTION FOR S^* IS

$$g_0 = V + \epsilon_N \frac{V''}{12} + \epsilon_N^2 \left[-\frac{V'^2}{24} + \frac{V^{(4)}}{240} \right]$$

$$g_1 = \frac{V''}{24} + \epsilon_N \frac{V^{(4)}}{480}$$

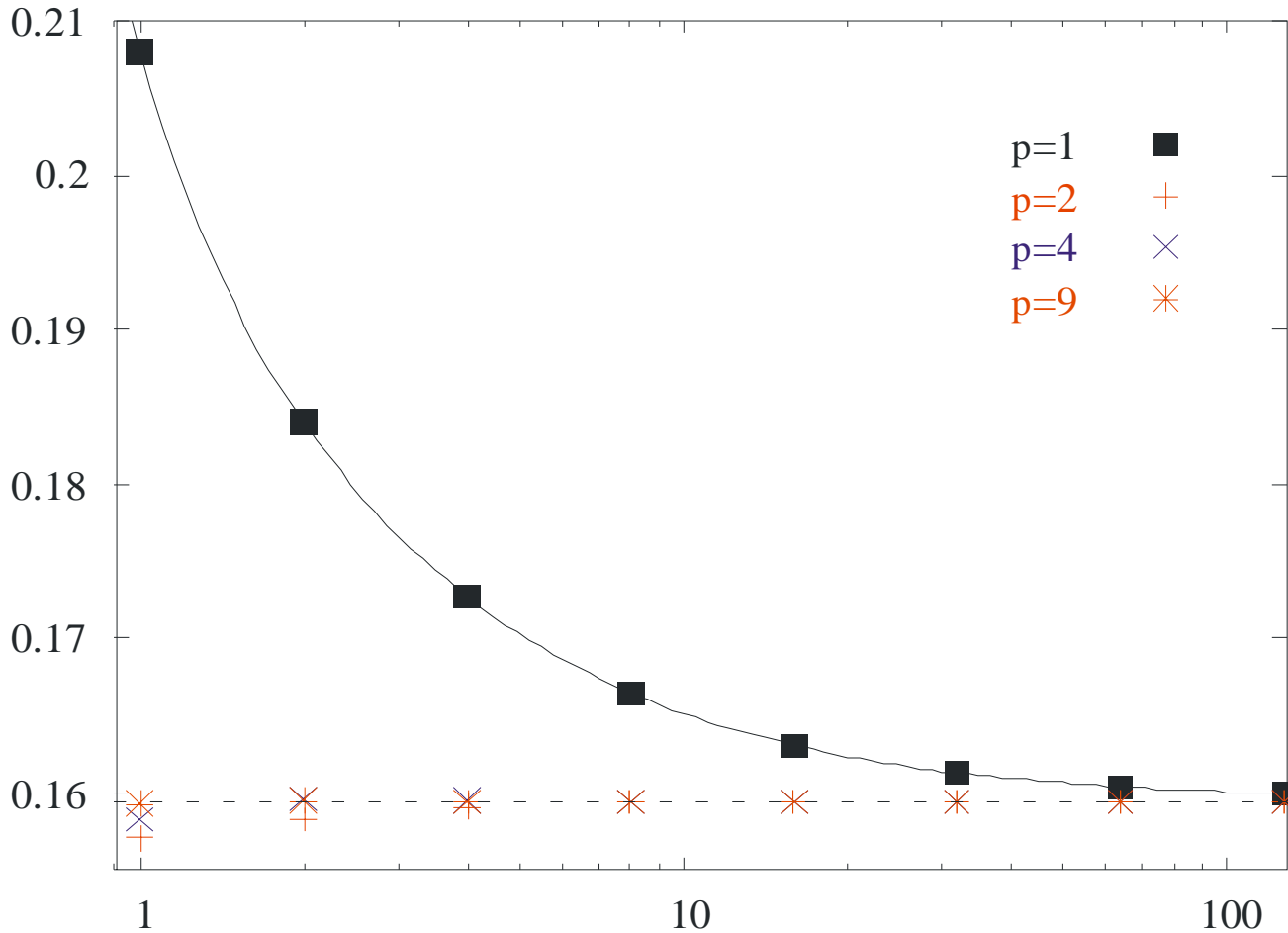
$$g_2 = \frac{V^{(4)}}{1920}$$

- SIMILARLY FOR HIGHER LEVELS p . NOTE THAT THE LEVEL p SOLUTION SATISFIES

$$A(a, b; T) = A_N^{(p)}(a, b; T) + O(1/N^p)$$

THIS IS THE EULER SUM FORMULA FOR PATH INTEGRALS.

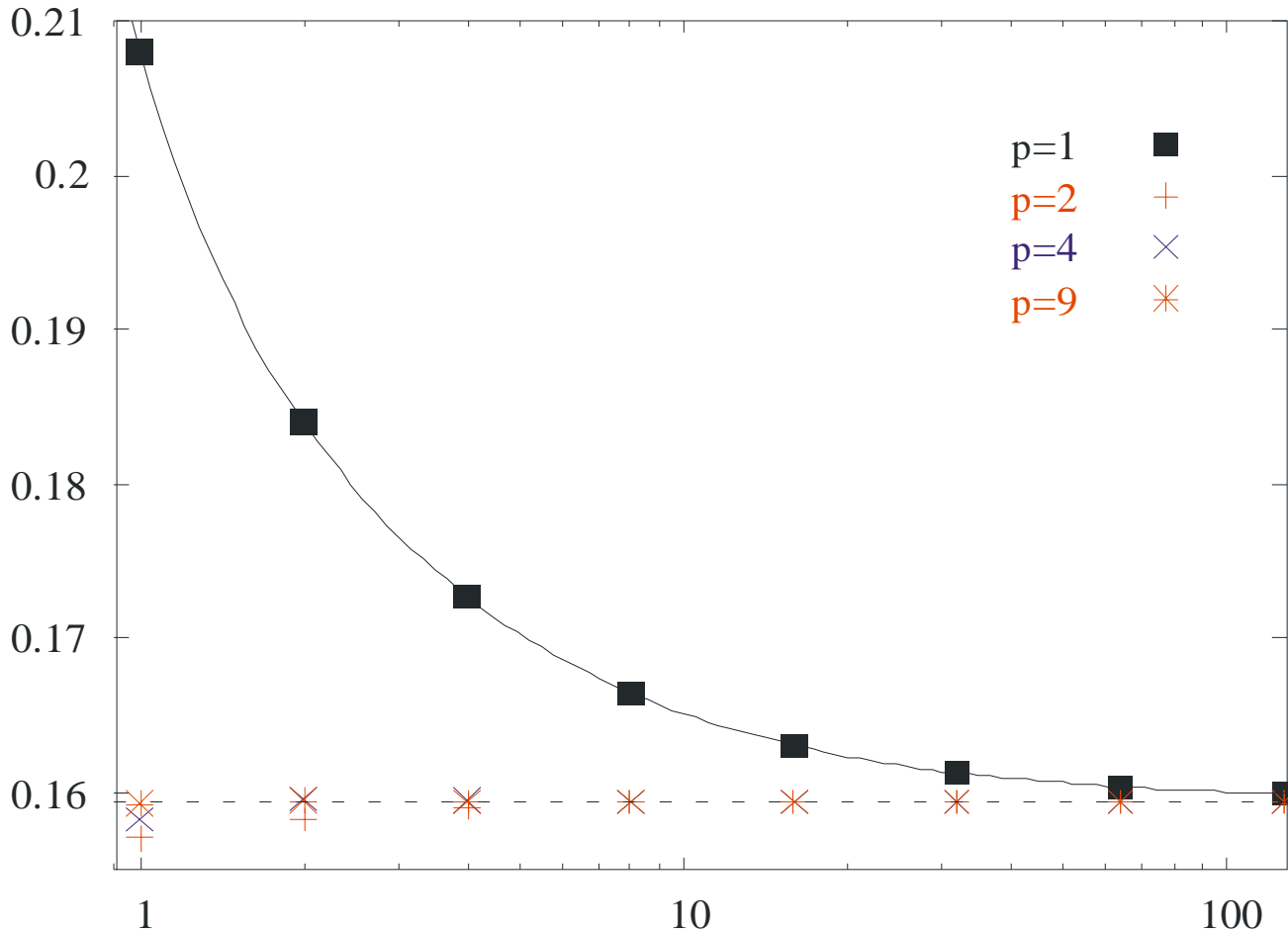
NUMERICAL RESULTS (1)



ANHARMONIC OSCILLATOR WITH QUARTIC COUPLING

$$g = 1, T = 1, a = 0, b = 1, N_{MC} = 10^7$$

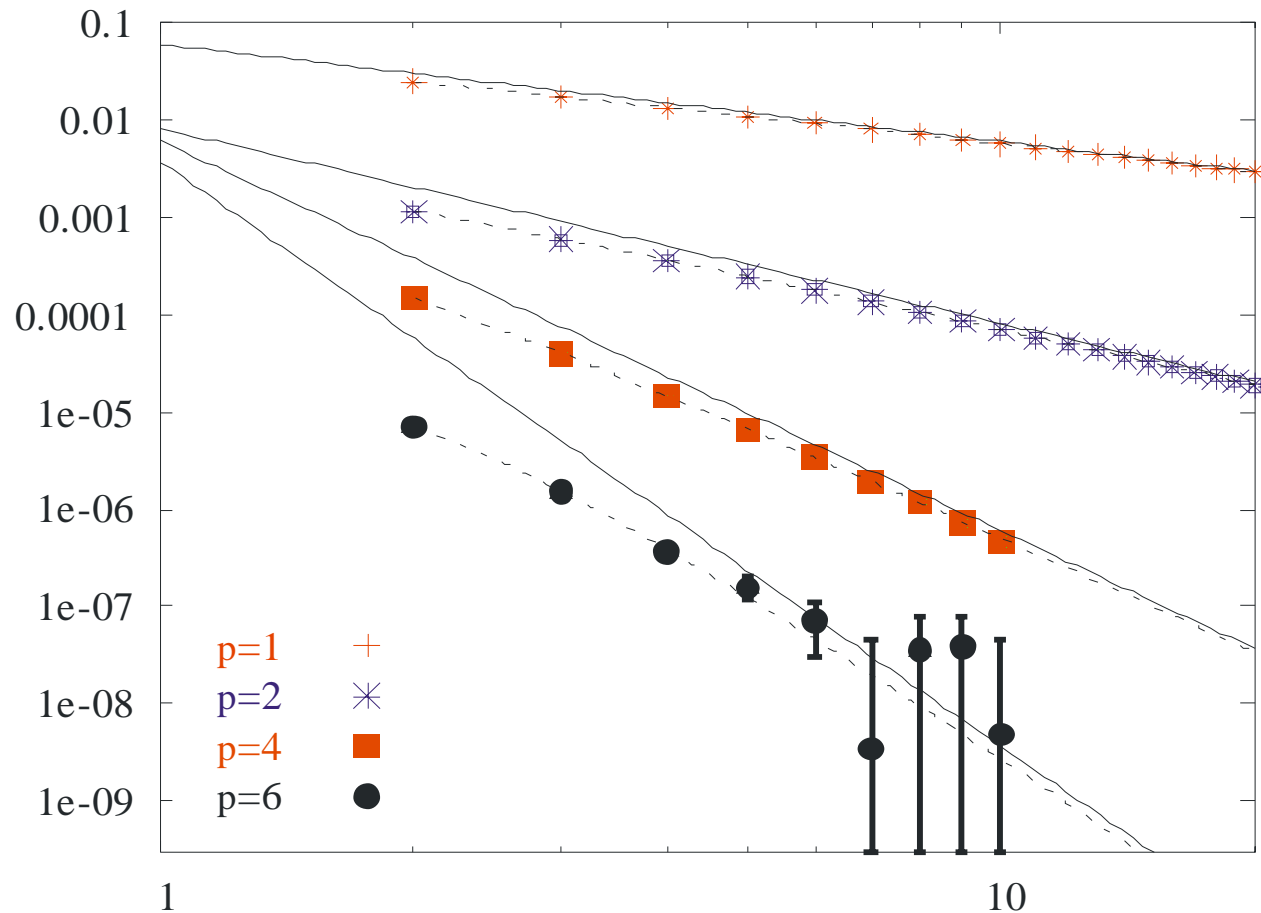
NUMERICAL RESULTS (2)



MODIFIED POESCHL-TELLER POTENTIAL

$$\alpha = 0.5, \beta = 1.5, T = 1, a = 0, b = 1, N_{MC} = 9.2 \cdot 10^7$$

NUMERICAL RESULTS (3)

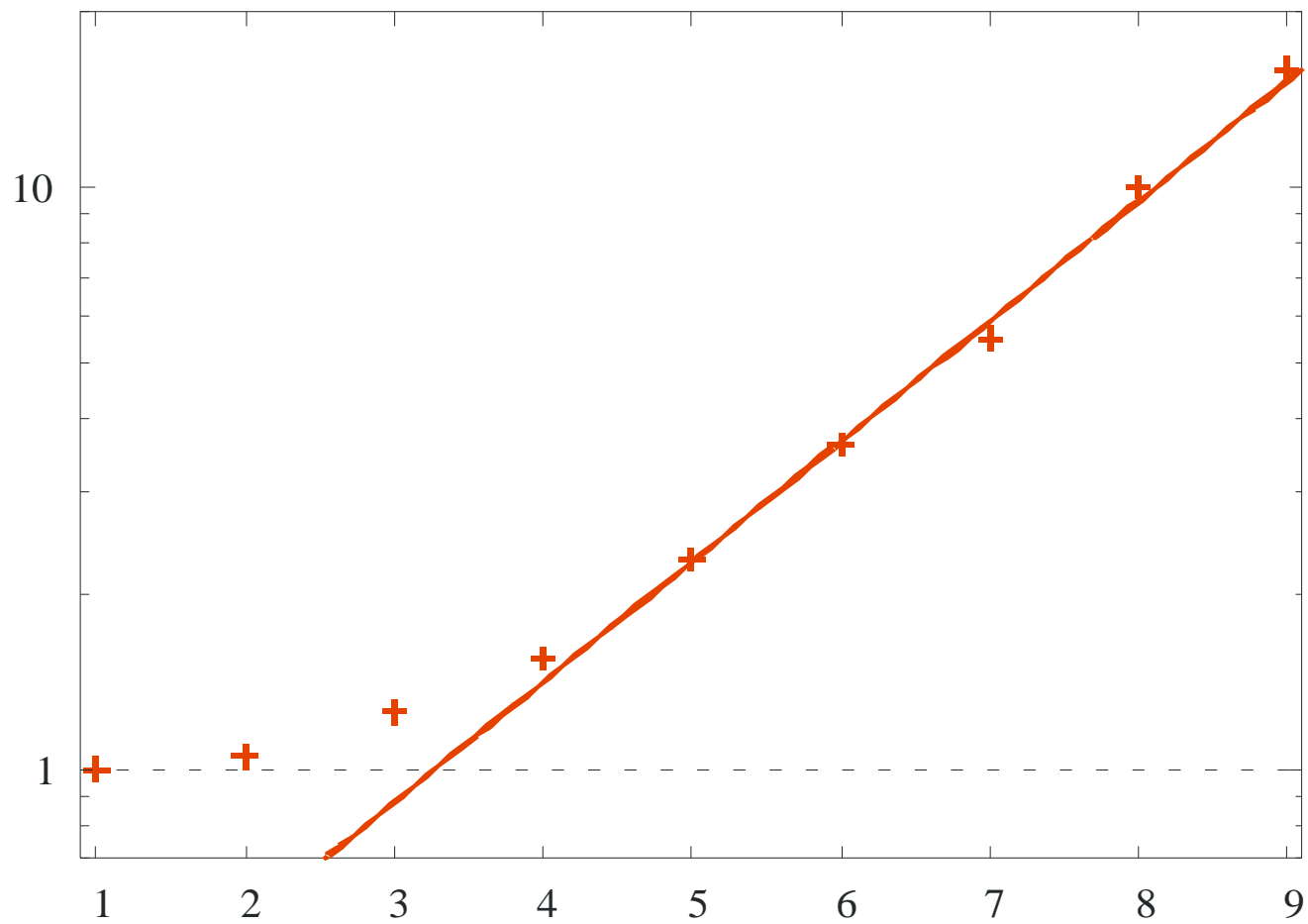


ERRORS FOR ANHARMONIC OSCILLATOR WITH QUARTIC COUPLING

$$g = 1, T = 1, a = 0, b = 1, N_{MC} = 9.2 \cdot 10^9 \quad (p = 1, 2),$$

$$N_{MC} = 9.2 \cdot 10^{10} \quad (p = 4), N_{MC} = 3.68 \cdot 10^{11} \quad (p = 6)$$

NUMERICAL RESULTS (4)



RELATIVE INCREASE IN COMPUTING TIME AS A FUNCTION OF p

DISCUSSION (1)

- THE GENERAL PATH INTEGRAL CALCULATION IS NOW SPEEDED UP BY MANY ORDERS OF MAGNITUDE.
- THE SAME PROCEDURE WORKS IN EUCLIDEAN AND MINKOWSKI FORMALISM (WITH APPROPRIATE $i\epsilon$ PRESCRIPTION).
- THE PROCEDURE HAS BEEN CHECKED ON SEVERAL DIFFERENT MODELS ACROSS A WIDE RANGE OF PARAMETERS (IN PERTURBATIVE AND NON-PERTURBATIVE REGIMES, ACROSS CRITICAL POINTS, ETC.)

DISCUSSION (2)

- EXTENSIONS TO MANY NON-RELATIVISTIC PARTICLES IN d DIMENSIONS AS WELL AS TO FIELD THEORIES IN $d > 1$ ARE IN PROGRESS.
- HIGHER DIMENSIONAL ANALOGUES OF THE INTEGRAL EQUATION ARE NOT A PROBLEM TO DERIVE. THE ASYMPTOTIC EXPANSION PROCEDURE ALSO WORKS IN ALL d . THE ALGEBRAIC RECURSIVE RELATIONS WILL BE MORE COMPLEX AND THIS COULD PRACTICALLY LIMIT US TO SMALLER VALUES OF p .

HIGHLIGHTS

- THIS IS AN ANALYTIC PROCEDURE THAT WORKS FOR A GENERIC MODEL.
- THE DIRECT OUTCOME OF THE PROCEDURE IS A SUBSTANTIAL SPEEDUP OF NUMERICAL CALCULATIONS OF PATH INTEGRAL (MILLION FOLD OR BETTER).
- IT NOW BECOMES POSSIBLE TO TACKLE COMPLEX MODELS THAT COULD NOT BE SOLVED PREVIOUSLY.
- AN IMPORTANT OUTCOME (NOT DISCUSSED IN THIS PRESENTATION) ARE NEW ANALYTICAL APPROXIMATION TECHNIQUES FOR PATH INTEGRALS.
- THE TRUE OUTCOME OF THE WORK IS HEURISTIC – I.E. IT LEADS TO AN INCREASE OF KNOWLEDGE ABOUT PATH INTEGRATION IN GENERAL. IT RAISES THE HOPE THAT, AS WITH ORDINARY INTEGRALS, THE DERIVATION OF EULER'S SUM FORMULA WILL BE A PRECURSOR TO THE SETTING UP OF A GENERAL THEORY OF PATH INTEGRATION.

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