Noncommutavity in the presence of dilaton field

B. Nikolić and B. Sazdović
Institute of Physics, Belgrade

Known results

- Dp-brane world-volume is non-commutative if open string ends on Dp-brane with Neveu-Schwarz field $B_{\mu\nu}$
  V. Schomerus, JHEP **06** *(1999) 030*
  F. Ardalan, H. Arfaei and M. M. Sheikh-Jabbari, JHEP **02** *(1999) 016; C. S. Chu and P. M. Ho, Nucl. Phys.* **B550** *(1999) 151.*
  N. Seiberg and E. Witten, JHEP **09** *(1999) 032.*
  F. Ardalan, H. Arfaei and M. M. Sheikh-Jabbari Nucl. Phys. **B576** *(2000) 578; C. S. Chu and P. M. Ho, Nucl. Phys.* **B568** *(2000) 447; T.Lee, Phys. Rev.* **D62** *(2000) 024022.*

- In the presence of linear dilaton field $\Phi = \Phi_0 + a_\mu x^\mu$ for $a^2 \neq 0$
  - Conformal part F of the world-sheet metric becomes a new non-commutative variable
  - Coordinate in the a_μ direction becomes commutative
    (Sazdović, hep-th/0408131)

New results

- $a^2 \neq 0$
  - We extend the space adding the new coordinate: the conformal part F of the world-sheet metric
  - The action in standard space with linear dilaton we can express in the extended space as dilaton free action
  - We can apply all known results for this case

- $a^2 = 0$
  - Primary first class constraint appears, which is generator of the local gauge symmetry
  - In terms of new **gauge invariant** and canonically conjugated variables
    * Action obtains the dilaton free form
    * String is described by effective coordinates $z^i = x^i + 2a^i F$

Method

- We apply canonical method
- We treat boundary conditions as constraints

Definition of the model

Action

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu} + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu} \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi R^{(2)} \right\}$$

- – $x^\mu(\xi)$ $(\mu = 0, 1, ..., D-1)$ space-time coordinates
  – ξ^α $(\alpha = 0, 1)$ world-sheet coordinates
  – $x^i(\xi)$ $(i = 0, 1, ..., p)$ Dp-brane coordinates

- Open string propagates in x^μ dependent background
  - metric tensor $G_{\mu\nu}$
  - antisymmetric tensor field $B_{\mu\nu} = -B_{\nu\mu}$
  - dilaton field Φ

- – Conformal gauge, $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta} \Longrightarrow R^{(2)} = 2\Delta F$
  – Chose for simplicity
    * $B_{\mu\nu} \to B_{ij}$ and $a_\mu \to a_i$
    * $G_{\mu\nu} = 0$ for $\mu = i \in \{0, 1, .., p\}$, $\nu = a \in \{p+1, ..., D-1\}$

$$S = S_1(x^a) + S_2(x^i, F)$$

$$S_1 = \kappa \int_\Sigma d^2\xi \frac{1}{2} \eta^{\alpha\beta} G_{ab} \partial_\alpha x^a \partial_\beta x^b$$

$$S_2 = \kappa \int_\Sigma d^2\xi \left[\left(\frac{1}{2} \eta^{\alpha\beta} G_{ij} + \epsilon^{\alpha\beta} B_{ij} \right) \partial_\alpha x^i \partial_\beta x^j + 2\eta^{\alpha\beta} a_i \partial_\alpha x^i \partial_\beta F \right]$$

Action depends on $F \longrightarrow$ no conformal invariance

Solution of the space-time field equations

- Conditions on world-sheet conformal invariance

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2 D_\mu a_\nu = 0$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho_{\mu\nu} - 2 a_\rho B^\rho_{\mu\nu} = 0$$

$$\beta^\Phi \equiv 4\pi\kappa \frac{D-26}{3} - R + \tfrac{1}{12} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - 4 D_\mu a^\mu + 4 a^2 = 0$$

$a_\mu = \partial_\mu \Phi$, $B_{\mu\rho\sigma}$ is field strength of the field $B_{\mu\nu}$

- Exact solution

$$G_{\mu\nu}(x) = G_{\mu\nu} = const\,, \quad B_{\mu\nu}(x) = B_{\mu\nu} = const$$

$$\Phi(x) = \Phi_0 + a_\mu x^\mu\,, \quad (a_\mu = const)$$

for $\quad a^2 = \kappa\pi \frac{26-D}{3}$

- $a^2 = 0 \quad \rightarrow \quad D = 26$ Critical string

- $a^2 > 0 \quad \rightarrow \quad D < 26$ Noncritical string

Extended space

- In extended ($D+1$ dim) space-time with the coordinates $y^A = (x^i, F)$

$$S_2 = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\eta^{\alpha\beta} G_{AB} + \varepsilon^{\alpha\beta} B_{AB}\right] \partial_\alpha y^A \partial_\beta y^B$$

where

$$G_{AB} = \begin{pmatrix} G_{ij} & 2a_i \\ 2a_j & 0 \end{pmatrix}, \qquad B_{AB} = \begin{pmatrix} B_{ij} & 0 \\ 0 & 0 \end{pmatrix}$$

- Same form as in the absence of dilaton field
  - In $D+1$ dimensions
  - With particular expressions for G_{AB} and B_{AB}

The case $a^2 \neq 0$

- For $a^2 \neq 0$ there exists inverse metric G^{AB}

$$G^{AB} = (G^{-1})^{AB} = \begin{pmatrix} P_T^{ij} & \frac{a^i}{2a^2} \\ \frac{a^j}{2a^2} & -\frac{1}{4a^2} \end{pmatrix}$$

where $P_{ij}^T = G_{ij} - \frac{a_i a_j}{a^2}$

- Apply standard procedure for dilaton free case on $D+1$ dimensional space-time with particular forms for G_{AB} and B_{AB}

$$\Pi_{\pm AB} = B_{AB} \pm \frac{1}{2} G_{AB} = \begin{pmatrix} \Pi_{\pm ij} & \pm a_i \\ \pm a_j & 0 \end{pmatrix}$$

$$(\Pi_{\pm}\Pi_{\mp})_{AB} = -\frac{1}{4} G^{eff}_{AB} = -\frac{1}{4} \begin{pmatrix} \tilde{G}_{ij} & 2a_i \\ 2a_j & 0 \end{pmatrix}$$

- Effective open string metrics
  * G^{eff}_{AB} in extended space-time
  * $\tilde{G}_{ij} \equiv (G - 4BP^T B)_{ij}$ in real space-time.

Canonical analysis

- Canonical variables $y^A = \{x^i, F\} \quad \pi_A = \{\pi_i, \pi\}$

- Currents on the Dp-brane

$$j_{\pm A} = \pi_A + 2\kappa \Pi_{\pm AB} y^{B'}$$

$$\{j_{\pm A}, j_{\pm B}\} = \pm 2\kappa G_{AB} \delta' \qquad \{j_{\pm A}, j_{\mp B}\} = 0$$

- Energy momentum tensor components

$$T_{\pm} = \mp \frac{1}{4\kappa} G^{AB} j_{\pm A} j_{\pm B}$$

  - Virasoro algebras

$$\{T_{\pm}, T_{\pm}\} = -[T_{\pm}(\sigma) + T_{\pm}(\bar{\sigma})]\delta', \qquad \{T_{\pm}, T_{\mp}\} = 0$$

- Canonical Hamiltonian $\quad \mathcal{H}_c = T_- - T_+$

Open string boundary conditions

- Lagrangian approach
  - Boundary conditions

$$\left(\frac{\partial S}{\partial x'^{\mu}}\delta x^{\mu} + \frac{\partial S}{\partial F'}\delta F\right)|_{\partial\Sigma} \equiv \frac{\partial S}{\partial y'^{A}}\delta y^{A}|_{\partial\Sigma} = 0$$

  - $y^A = \{x^i,\, F\} \longrightarrow$ Neumann boundary conditions arbitrary variations δx^i and δF on the string endpoints

$$\gamma_A^{(0)}|_{\partial\Sigma} = 0$$

$$\gamma_A^{(0)} \equiv \frac{\delta S}{\delta y'^A} = \kappa(-G_{AB}y'^B + 2B_{AB}\dot{y}^B)$$

  - $x^a \longrightarrow$ Dirichlet boundary conditions fixed edges of the string $\quad \delta x^a|_{\partial\Sigma} = 0$
  - In terms of the currents

$$\gamma_A^{(0)} = \gamma_{A-} + \gamma_{A+}\,, \qquad \gamma_{A\pm} \equiv \Pi_{\mp AB} j_{\pm}^{B}$$

- Hamiltonian approach
  - Hamiltonian is a generator of the time translations
  - it must be differentiable in coordinates and momenta

$$\delta H_c = \delta H_c^{(R)} - \gamma_A^{(0)}\delta y^A\Big|_0^{\pi} \qquad (1)$$

Constraints

- Consider $\gamma_A^{(0)}|_{\partial\Sigma}$ as a constraint

- Using Poisson brackets

$$\{H_c, j_{\pm A}\} = \mp j'_{\pm A}$$

  - Diarc consistency conditions generate infinite set $\gamma_A^{(n)}|_{\partial\Sigma} = 0 \quad (n \geq 1)$,

$$\gamma_A^{(n)} \equiv \{H_c, \gamma_A^{(n-1)}\} = \partial_\sigma^n \{\gamma_{A-} + (-1)^n \gamma_{A+}\}$$

  - In compact form
    - at $\sigma = 0$

$$\Gamma_A(\sigma) \equiv \sum_{n\geq 0} \frac{\sigma^n}{n!} \gamma_A^{(n)}(0) = \gamma_{A-}(\sigma) + \gamma_{A+}(-\sigma)$$

    - at $\sigma = \pi$

$$\bar{\Gamma}_A(\sigma) \equiv \sum_{n\geq 0} \frac{(\sigma-\pi)^n}{n!} \gamma_A^{(n)}(\pi) = \gamma_{A-}(\sigma) + \gamma_{A+}(2\pi - \sigma)$$

- Periodicity
  - Only arguments of positive chirality currents differ
    $\Rightarrow \gamma_{i+}(-\sigma) = \gamma_{i+}(2\pi - \sigma) \qquad \gamma_{+}(-\sigma) = \gamma_{+}(2\pi - \sigma)$
  - All currents and all variables
    $y^A = \{x^i, F\}$ and $\pi_A = \{\pi_i, \pi\}$
    are periodic in σ: $\quad \sigma \to \sigma + 2\pi$

- All constraints weakly commute with hamiltonian

$$\{H_c, \Gamma_A(\sigma)\} = \Gamma'_A(\sigma)$$

  there are no more constraints

- Algebra of constraints

$$\{\Gamma_A(\sigma), \Gamma_B(\bar{\sigma})\} = -\kappa G^{eff}_{AB} \delta'(\sigma - \bar{\sigma})$$

  were we introduced effective metric tensor in extended space

$$G^{eff}_{AB} \equiv G_{AB} - 4 B_{AC} G^{CD} B_{DB}$$

  - We will refer to it as the **open string metric tensor**, the metric tensor seen by the open string

- For $\quad \det G^{eff}_{AB} \neq 0$
  all constraints are of the second class

Solution of the boundary conditions

- Introduce Dirac brackets or **solve constraints**
- Open string variables $(q^A,\, p_A)$ $(\bar{q}^A,\, \bar{p}_A)$

$$q^A(\sigma) = \frac{1}{2}\left[y^A(\sigma) + y^A(-\sigma)\right], \quad \bar{q}^A(\sigma) = \frac{1}{2}\left[y^A(\sigma) - y^A(-\sigma)\right]$$
$$p_A(\sigma) = \frac{1}{2}\left[\pi_A(\sigma) + \pi_A(-\sigma)\right], \quad \bar{p}_A(\sigma) = \frac{1}{2}\left[\pi_A(\sigma) - \pi_A(-\sigma)\right]$$

- The constraints in terms of open string variables

$$\Gamma_A(\sigma) = 2(BG^{-1})_A{}^B p_B + \bar{p}_A - \kappa G^{eff}_{AB} \bar{q}'^B$$

  - Symmetric and antisymmetric parts vanish separately
  - Antisymmetric (bar) variables in terms of symmetric ones

$$\bar{p}_A = 0\,, \qquad \bar{q}'^A = \frac{2}{\kappa}(G^{-1}_{eff} B G^{-1})^{AB} p_B$$

Effective theory

- The original variables in terms of new ones

$$y^A = q^A + \frac{2}{\kappa}(G^{-1}_{eff} B G^{-1})^{AB} \int^\sigma d\sigma_1 p_B, \quad \pi_A = p_A$$

- Effective energy-momentum tensor in terms of new variables

$$T_\pm[y^A(q^A, p_A), \pi_A(p_A)] = \tilde{T}_\pm(q^A, p_A)$$

– has exactly the same form as $T_\pm$

$$\tilde{T}_\pm = \mp \frac{1}{4\kappa} G^{AB}_{eff} \tilde{j}_{\pm A} \tilde{j}_{\pm B}$$

∗ but in **new background**

$$G_{AB} \to G^{eff}_{AB} = G_{AB} - 4(BG^{-1}B)_{AB}\,,$$
$$B_{AB} \to \tilde{B}_{AB} = 0 \qquad (2)$$

∗ or in components

$$G_{ij} \to \tilde{G}_{ij} = G_{ij} - 4B_{ik}P^{Tkq}B_{qj}\,,$$
$$B_{ij} \to \tilde{B}_{ij} = 0\,, \qquad \Phi \to \tilde{\Phi} = \Phi_0 + a_i q^i$$

Non-commutativity in presence of dilaton

- Separate the center of mass, $y^A_{cm} = \frac{1}{\pi}\int_0^\pi d\sigma y^A(\sigma)$
  $y^A(\sigma) = y^A_{cm} + Y^A(\sigma)$

$$\{Y^A(\sigma), Y^B(\bar\sigma)\} = 2\Theta^{AB}\Delta(\sigma+\bar\sigma)$$

$$\Theta^{AB} = -\frac{1}{\kappa}(G^{-1}_{eff}BG^{-1})^{AB} = \begin{pmatrix} \Theta^{ij} & \Theta^i \\ -\Theta^j & 0 \end{pmatrix}$$

$$\Theta^{AB} = -\Theta^{BA}$$

$$\Delta(\sigma+\bar\sigma) \equiv \begin{cases} -1 & \sigma = 0 = \bar\sigma \\ 1 & \sigma = \pi = \bar\sigma \\ 0 & \text{otherwise} \end{cases}$$

- In the component form

$$\{X^i(\sigma), X^j(\bar\sigma)\} = 2\Theta^{ij}\Delta(\sigma+\bar\sigma)$$

$$\Theta^{ij} = -\frac{1}{\kappa}(\tilde P^T B P^T)^{ij} \qquad \Theta^{ij} = -\Theta^{ji}$$

- – New non-commutative variable (conformal factor)

$$\{X^i(\sigma), F(\bar\sigma)\} = 2\Theta^i \Delta(\sigma + \bar\sigma)$$

$$\Theta^i = \frac{(aB\tilde{G}^{-1})^i}{2\kappa a^2} = \frac{(\tilde{a}B)^i}{2\kappa\tilde{a}^2}$$

  – and one commutative Dp-brane direction $x \equiv a_i x^i$

$$a_i\Theta^{ij} = 0 \qquad a_i\Theta^i = 0$$

$$\Rightarrow \qquad \{x(\sigma), x^j(\bar\sigma)\} = 0\,, \qquad \{x(\sigma), F(\bar\sigma)\} = 0$$

  – Total number of non-commuting coordinates remains the same

The case $a^2 = 0$

$$\det G_{AB} = -4a^2 \det G_{ij}$$

for $a^2 = 0$ $\quad G_{AB}$ is singular (doesn't have inverse)

- There is a constraint in the theory

$$j = a^i j_{\pm i} - \frac{1}{2} j_{\pm D} = a^i \pi_i - \frac{1}{2}\pi + 2\kappa a^i B_{ij} x'^j$$

- Canonical hamiltonian

$$\mathcal{H}_c = \frac{1}{2\kappa}\pi_i^2 + \frac{\kappa}{2} x' G_{eff} x' + 2\pi B x' + 2\kappa a_i x^{i'} F'$$

- If we define currents in the form

$$\dot{x}^i + 2a^i \dot{F} = \frac{G^{ij}}{2\kappa}(j_{+j} + j_{-j})$$

$$x^{i'} + 2a^i F' = \frac{G^{ij}}{2\kappa}(j_{+j} - j_{-j})$$

Then hamiltonian is described in terms of new currents

$$\mathcal{H}_c = T_- - T_+ \qquad T_\pm = \mp \frac{1}{4\kappa} G^{ij} j_{\pm i} j_{\pm j}$$

in the same form as before

- Total hamiltonian

$$\mathcal{H}_T = \mathcal{H}_c + \lambda j$$

- j is a first class constraint

$$\{\mathcal{H}_T, j\} = 0$$

- Generator of the symmetry

$$\delta_\eta X = \{X, G\} \qquad G \equiv \int d\sigma \eta(\sigma) j(\sigma)$$

  - Gauge transformations

$$\delta_\eta x^i = a^i \eta \qquad \delta_\eta F = -\frac{1}{2}\eta$$

$$\delta_\eta \pi_i = 2\kappa a^j B_{ji} \eta' \qquad \delta_\eta \pi = 0$$

  - New variables

$$z^i = x^i + 2a^i F \qquad P_i = \pi_i + 4\kappa F' a^j B_{ji}$$

    * gauge invariant

$$\delta_\eta z^i = 0 \qquad \delta_\eta P_i = 0$$

    * canonically conjugated

$$\{z^i, P_j\} = \delta^i_j \delta(\sigma - \overline{\sigma})$$

- In terms of new variables z^i and P_i
  - hamiltonian

$$\mathcal{H}_c(\pi_i, x^i, F) = \mathcal{H}_c(P_i, z^i, 0)$$

  - and currents

$$j_{\pm i} = \pi_i + 2\kappa\Pi_{\pm ij}x'^j \pm 2\kappa a_i F' = P_i + 2\kappa\Pi_{\pm ij}z'^j$$

  have the form of the starting one but without F field

- Non-commutativity relations for new variables z^i, are the same as in the case $\Phi = 0$

$$\left\{z^i(\tau,\sigma), z^j(\tau,\overline{\sigma})\right\} = 2\Theta^{ij}\Delta(\sigma+\overline{\sigma}) \qquad (3)$$

$$\Theta^{ij} = \frac{-1}{\kappa}(G_{eff}^{-1}BG^{-1})^{ij} \qquad (4)$$

  - The case $a^2 = 0$ with z^i is equivalent to the case $\Phi = 0$ with x^i including the number of space-time dimensions, $D = 26$