

# Regularization of Gauge Theory on Noncommutative $\mathbb{R}^4$

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# Outline

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- Noncommutativity
- Gauge Theory on  $\mathbb{R}_\theta^4$
- Regularization of the Gauge Theory
- Instantons
- Conclusions



# Noncommutativity

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- Infinities in QFT suggest new structures beyond
- Noncommutativity

$$[x_i, x_j] = i\theta_{ij}$$

as a model of quantized spacetime

- In the canonical case  $\theta_{ij} \in \mathbb{R}$  singularities are not cured
- Regularization of the noncommutative theory necessary



# Canonical Noncommutative $\mathbb{R}^4$

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- In the Euclidian, the coordinates can be made to form two Heisenberg algebras

$$[x_L^+, x_L^-] = \theta, \quad [x_R^+, x_R^-] = \theta, \quad [x_L^\pm, x_R^\pm] = 0$$

- Representation as creator and annihilator operators on usual Fock space
- Derivatives are inner:  $\partial_i \hat{=} -\frac{i}{\theta}[x_i, \cdot]$



# Gauge Theory

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- Matrix action with ground state  $X_i = x_i$

$$S_\theta = -\frac{(2\pi)^2}{2g^2\theta^2} \text{tr} ([X_i, X_j] - i\theta_{ij})^2$$

- Covariant coordinates  $X_i = x_i + A_i$

$$X_i \rightarrow UX_iU^\dagger$$

$$A_i \rightarrow U[x_i, U^\dagger] + UA_iU^\dagger$$

- Field strength

$$iF_{ij} = [X_i, X_j] - i\theta_{ij} = [x_i, A_j] - [x_j, A_i] + [A_i, A_j]$$



# Problems

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- Rank of the gauge group not fixed
- Infinities of the commutative theory persist
- New infinities: IV/UR-mixing

→ **Regularization** necessary



# The Fuzzy Sphere

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For the regularization, we need a finite-dimensional representation, that in some limit goes to  $\mathbb{R}_\theta^4$ . The fuzzy sphere  $S_M^2$  is used as a building block:

- creators  $\lambda_i$  with  $i \in \{1, 2, 3\}$  and relations

$$[\lambda_i, \lambda_j] = i\epsilon_{ijk}\lambda_k \quad \text{and} \quad \sum_i \lambda_i \lambda_i = \frac{M^2 - 1}{4}$$

- $M$ -dim. Irreps of  $su(2)$
- Analoqs of spherical harmonics up to an angular momentum  $M - 1$ .
- Coordinates  $x_i$  correspond to  $x_i \approx \frac{2R}{M}\lambda_i$ , tangential derivatives  $J_i = [\lambda_i, \cdot]$



# A four-dimensional space

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- To go to four dimensions, we simply take two copies of the fuzzy sphere
- This new space  $S_M^2 \times S_M^2$  is generated by the  $M^2$ -dimensional matrices

$$\lambda_{iL} = \lambda_i \otimes 1 \quad \text{and} \quad \lambda_{iR} = 1 \otimes \lambda_i$$

- Coordinates scale again as  $x_{iL/R} \approx \frac{2R}{M} \lambda_{iL/R}$ , the tangential derivatives as  $J_{iL/R} = [\lambda_{iL/R}, \cdot]$





# Gauge Theory on $S_M^2 \times S_M^2$

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To do gauge theory, we use an action over six  $M^2$ -dimensional matrices  $B_\mu = (B_{iL}, B_{iR})$

$$S_{fuzzy} = \frac{8\pi^2}{M^2} \text{tr}((i[B_{iL/R}, B_{jL/R}] + \epsilon_{ijk} B_{kL/R})^2 - [B_{iL}, B_{jR}]^2 + V(B))$$

- The ground state obviously is  $\lambda_{iL/R}$
- The covariant coordinates  $B_{iL/R} = \lambda_{iL/R} + A_{iL/R}$  form the gauge theory
- The potential  $V(B) = 2(B_{iL}B_{iL} - \frac{M^2-1}{4})^2 + 2(B_{iR}B_{iR} - \frac{M^2-1}{4})^2$  stabilizes the size of the representation



# *Finiteness of the model*

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- The gauge group is fixed to be  $U(1)$
- Gauge groups  $U(N)$  are implemented by changing the size of the matrices to  $NM^2$
- Quantization can be performed via a path integral over the matrix entries

$$Z[J] = \int dB_\mu e^{-S[B_\mu] + \text{tr} B_\mu J_\mu}.$$

It is well defined and finite for every  $M$



# Scaling limit to $\mathbb{R}_\theta^4$

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- The north pole (i.e.  $\lambda_3 \approx \frac{M}{2}$ ) of  $S_M^2$  is blown up by a double scaling limit  $R, M \rightarrow \infty$ , keeping  $\frac{2R^2}{M} = \theta$  fixed

$$[x_1, x_2] = i \frac{2R}{M} \sqrt{R^2 - x_1^2 - x_2^2} = i\theta + O(1/M).$$

This is nothing but  $\mathbb{R}_\theta^2$

- We get the coordinates  $x_i$  with  $i \in \{1, \dots, 4\}$  of  $\mathbb{R}_\theta^4$  by scaling  $\lambda_{iL}$  and  $\lambda_{iR}$  as

$$\sqrt{\frac{2\theta}{M}} \lambda_{1,2L} \rightarrow x_{1,2} \quad \text{and} \quad \sqrt{\frac{2\theta}{M}} \lambda_{1,2R} \rightarrow x_{3,4}$$



# Scaling limit of the gauge theory

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- Under the double scaling limit, the covariant coordinates  $B_\mu$  on  $S_M^2 \times S_M^2$  are mapped to the covariant coordinates  $X_\mu$  on  $\mathbb{R}_\theta^4$

$$\sqrt{\frac{2\theta}{M}} B_{1,2/,L} \rightarrow X_{1,2} \quad \text{and} \quad \sqrt{\frac{2\theta}{M}} B_{1,2/,R} \rightarrow X_{3,4}$$

- The complete gauge theory from  $S_M^2 \times S_M^2$  is mapped to  $\mathbb{R}_\theta^4$ , we get

$$S_{fuzzy} \rightarrow S_\theta$$

- This is the desired **regularization** for gauge theory on  $\mathbb{R}_\theta^4$



# Instantons

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- On  $\mathbb{R}_\theta^4$ , a class of instantons can be constructed by setting

$$X_i = \begin{pmatrix} \text{diag}(c_{1,i}, \dots, c_{k,i}) & 0 \\ 0 & x_i \end{pmatrix},$$

fulfilling the EOMs  $[X_i, F_{ij}] = 0$

- On  $S_M^2 \times S_M^2$ , we can mimic this by setting

$$B_{iL} = \alpha \begin{pmatrix} \text{diag}(d_{1,Li}, \dots, d_{k,Li}) & 0 \\ 0 & \lambda'_i \otimes 1_{M'' \times M''} \end{pmatrix},$$
$$B_{iR} = \alpha \begin{pmatrix} \text{diag}(d_{1,Ri}, \dots, d_{k,Ri}) & 0 \\ 0 & 1_{M' \times M'} \otimes \lambda''_i \end{pmatrix},$$

fulfilling the EOMs on  $S_M^2 \times S_M^2$



# Instantons

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- On  $\mathbb{R}_\theta^4$ , we could have any charge  $k$  of the instantons
- On  $S_M^2 \times S_M^2$ , the overall size of the the matrix  $B_\mu$  is fixed to  $M^2$ . For finite instantons, we need

$$M' = M - l, \quad M'' = M + l \quad \text{and} \quad k = l^2$$

- We get back the instantons on  $\mathbb{R}_\theta^4$ , but the regularization produces a superselection rule for the instanton charge



# Conclusions

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- Gauge theory on  $\mathbb{R}_\theta^4$  must be regularized (because of infinities and sectors for every rank of the gauge group) to be well defined
- Gauge theory on  $S_M^2 \times S_M^2$  provides such a regularization
- We were able to match parts of the instanton sectors on  $S_M^2 \times S_M^2$  and  $\mathbb{R}_\theta^4$
- The regularization introduces a superselection rule for the instanton charge

