Difference Discrete and Fractional Variational Principles: Present and Perspectives

Dumitru Baleanu
Department of Mathematics and Computer sciences
Faculty of Arts and Sciences
Çankaya University, Ankara, Turkey
and
Institute of Space Sciences
P.O. Box, MG-23, R 76900
Magurele-Bucharest, Romania

Part I

Difference Discrete Variational Principles

1. The Difference Operator
2. Linear Difference Equations
3. Linear Difference Systems
4. Discrete Variational Principles

The Difference Operator

- Let $y(t)$ be a function of a real or complex variable t.

The difference operator Δ is defined by

- $$\Delta y(t) = y(t+1) - y(t)$$

- $h > 0$, $z(s+h) - z(s)$.

Let $y(t) = z(th)$.

Then,

$$z(s+h) - z(s) = z(th+h) - z(th)$$
$$= y(t+1) - y(t) = \Delta y(t)$$

Example

- $\Delta_t te^n = (t+1)e^n - te^n = e^n$
- $\Delta_n te^n = te^{n+1} - te^n = te^n(e-1)$
- $\Delta^2 y(t) = \Delta(\Delta y(t))$

$$= \Delta(y(t+1) - y(t))$$
$$= (y(t+2) - y(t+1)) - (y(t+1) - y(t))$$
$$= y(t+2) - 2y(t+1) + y(t)$$

References

1. Difference Equations
An Introduction with
Applications,
W. G. Kelley and A. C. Peterson
Academic Press, inc. , New York, 1991

2. T. D. Lee, Phys. Lett. B 122, (1983) 217.

3. T. D. Lee, J. Statis. Phys. 46 (1987) 843

4. J. M. Sanz-Serna and M. P. Calvo,
"Numerical Hamiltonian Problems",
Chapman and Hall, London (1994).

5. J. Moser and A. P. Veselov, Comm. Math.
Phys. 139, (1991) 217.

6. L. X. Dong, G. H. Ying, L. Y. Qi and
W. Ke, Comm. Theor. Phys. 42
(2004), 443.

7. H. Y. Guo and Ke Wu,
J. Math. Phys. **44** (2003), 5978.

8. C. D. Bartolo, R. Gambini, R. Porto and J. Pullin, J. Math. Phys. **46** (2005), 012901

9. R. Gambini and J. Pullin, Phys. Rev. Lett. 94, 101302 (2005)

10. R. Gambini and J. Pullin, Class. Quant. Grav. 20, 3341 (2003)

11. R. Gambini and J. Pullin, Phys. Rev. Lett. 93, 240401 (2004)

12. R. Gambini and J. Pullin, int. J. Mod. Phys. D 12 (2003)

13. R. Gambini and J. Pullin, Phys. Rev. Lett. 90, 021301 (2003).

14. J. Fahd, D. Baleanu, in press Physica Scripta (2005).

Definition

The "binomial coefficient" $\binom{t}{r}$ is defined by

$$\binom{t}{r} = \frac{t^{(r)}}{\Gamma(r+1)}$$

Theorem

whenever $t^{(r)}$ is defined, we have

a) $\Delta_t t^{(r)} = r t^{(r-1)}$

b) $\Delta_t \binom{t}{r} = \binom{t}{r-1}, \ r \neq 0$

Example :

Find a solution to the difference equation

$$y(t+2) - 2y(t+1) + y(t) = t(t-1) \quad (1)$$

- (1) can be written as

$$\Delta^2 y(t) = t^{(2)}$$

$$\Downarrow$$

$y(t) = \frac{t^{(4)}}{12}$ is a solution of (1).

Definition

The "factorial function" $t^{(n)}$ is defined as follows:

a) if $n = 1, 2, 3 \ldots$, then $t^{(n)} = t(t-1)(t-2)\ldots(t-n+1)$

b) if $n = 0$, then $t^{(0)} = 1$

c) if $n = -1, -2, -3, \ldots$, then

$$t^{(n)} = \frac{1}{(t+1)(t+2)\ldots(t-n)}$$

d) if n is not integer, then

$$t^{(n)} = \frac{\Gamma(t+1)}{\Gamma(t-n+1)}$$

Remark

$$\frac{\Gamma(t+1)}{\Gamma(t-n+1)} = \frac{t\Gamma(t)}{\Gamma(t-n+1)} = \frac{t(t-1)\Gamma(t-1)}{\Gamma(t-n+1)} = \ldots =$$

$$= t(t-1)\ldots(t-n+1)\frac{\Gamma(t-n+1)}{\Gamma(t-n+1)} =$$

$$= t(t-1)\ldots(t-n+1).$$

$$\left(\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt\right)$$

- The shift operator E is defined by

$$E\,y(t) = y(t+1)$$

- $I\,y(t) = y(t)$

- $\Delta = E - I$

- $\Delta^n y(t) = (E - I)^n y(t)$

$$= \sum_{k=0}^{n} \binom{n}{k} (-I)^n E^{n-k} y(t)$$

- $E^n y(t) = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} y(t)$

<u>Theorem</u>

a) $\Delta^m(\Delta^n y(t)) = \Delta^{m+n} y(t)$ for all positive integers m and n

b) $\Delta(y(t) + z(t)) = \Delta y(t) + \Delta z(t)$

c) $\Delta(C y(t)) = C\Delta y(t)$, C is a constant

d) $\Delta(y(t) z(t)) = y(t)\,\Delta z(t) + E(z(t))\,\Delta y(t)$

e) $\Delta\left(\dfrac{y(t)}{z(t)}\right) = \dfrac{z(t)\,\Delta y(t) - y(t)\,\Delta z(t)}{z(t)\,E\,z(t)}$

- <u>Definition</u>

An indefinite sum (or antidifference) of $y(t)$ denoted $\sum y(t)$ is any function so that

$$\Delta\left(\sum y(t)\right) = y(t)$$ for all t in the domain of y.

<u>Ex.</u>

Calculate the indefinite sum $\sum 6^t$

$$\Delta 6^t = 6^{t+1} - 6^t = (6-1)6^t = 5\,6^t$$

$$\Delta \frac{6^t}{5} = 6^t$$

The most general solution is

$$\sum 6^t = \frac{6^t}{5} + C(t),$$ such that

$$\Delta C(t) = 0.$$

<u>Theorem</u>

if $z(t)$ is an indefinite sum of $y(t)$, then every indefinite sum of $y(t)$ is given by

$$\sum y(t) = z(t) + C(t)$$

where $C(t)$ has the same domain as y and $\Delta C(t) = 0$

The z-transform

Definition

The z-transform of a sequence $\{y_k\}$ is a function $Y(z)$ of a complex variable defined by

$$Y(z) = Z\{y_k\} = \sum_{k=0}^{\infty} \frac{y_k}{z^k}$$

for those values of z for which the series converges.

Example: Find the z-transform of $\{y_k = 1\}$

$$Y(z) = Z(1) = \sum_{k=0}^{\infty} \frac{1}{z^k} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}, \ |z| > 1$$

Example:
Find the z-transform of the sequence $\{u_k = a^k\}$

$$Y(z) = Z(a^k) = \sum_{k=0}^{\infty} \frac{a^k}{z^k} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k = \frac{1}{1 - \frac{a}{z}}$$

$$= \frac{z}{z-a}, \ |z| > |a|.$$

Linear Difference Equations

- First order Equations

$$y(t+1) - p(t)y(t) = r(t) \qquad (*)$$

$p(t) \neq 0$, $\forall t$.

- if $p(t) = 1$, $\forall t$, <u>then</u>

$$\Delta y(t) = r(t)$$

$$\Downarrow$$

$$y(t) = \sum r(t) + C(t) \quad , \quad \Delta C(t) = 0$$

Let's assume the domain of interest is a discrete set $t = a, a+1, a+2 \cdots$

<u>step 1.</u> Homogeneous equation

$$u(t+1) = p(t)\,u(t)$$

By iteration

$$u(a+1) = p(a)\,u(a)$$

$$u(a+2) = p(a+1)\,p(a)\,u(a),$$

$$\vdots$$

$$u(a+n) = u(a) \prod_{k=0}^{n-1} p(a+k)$$

or

$$u(t) = u(a) \prod_{s=a}^{t-1} p(s) \quad , (t = a, a+1 \cdots)$$

By substituting $y(t) = u(t)\, v(t)$ into (*) where v is to be determined.

$$u(t+1)\, v(t+1) - p(t)\, u(t) v(t) = r(t)$$

or

$$v(t) = \sum \frac{r(t)}{E u(t)} + C.$$

so

$$y(t) = u(t) \left[\sum \frac{r(t)}{E u(t)} + C \right]$$

Discrete variational principles.

- Classical mechanics

$$L = \frac{1}{2}(\dot{\vec{r}})^2 - V(r) \qquad (1)$$

$$S = \int_a^b L\,dt$$

$dt \to \Delta t_n$

$\vec{r}(t_0) = \vec{r}_0$

$\vec{r}(t_f) = \vec{r}_f$

in the discrete time formalism we replace the continuous function $\vec{r}(t)$ by a sequence of discrete values,

$(\vec{r}_0, t_0), (\vec{r}_1, t_1), \ldots, (\vec{r}_N, t_N), (\vec{r}_{N+1}, t_{N+1})$,

where $(\vec{r}_{N+1}, t_{N+1}) = (\vec{r}_f, t_f)$

The action (1) is then replaced by

$$A = \sum_{n=1}^{N+1} \left[\frac{1}{2} \frac{(\vec{r}_n - \vec{r}_{n-1})^2}{t_n - t_{n-1}} \right.$$

$$\left. - \frac{1}{2}(t_n - t_{n-1}) \left[V(\vec{r}_n) + V(\vec{r}_{n-1}) \right] \right]$$

with $t_n > t_{n-1}$.

Newton's equation of motion is derived by setting the derivative

$$\frac{\partial A}{\partial \vec{r}_N} = 0$$

Keeping the initial and final configurations fixed, we have altogether N such equations

$$\vec{v}_{n+1} - \vec{v}_n = -\frac{1}{2}(t_{n+1} - t_{n-1})\,\vec{\nabla} V(\vec{r}_n)$$

with

$$\vec{v}_n = \frac{\vec{r}_n - \vec{r}_{n-1}}{t_n - t_{n-1}}$$ is the velocity.

For any given time distribution $t_1, \dots, t_N$, the positions, $\vec{r}_1, \dots, \vec{r}_N$ can be determined.

In discrete mechanics we require that the time distribution should also be determined by the same action!

$\Downarrow$

t_n is treated as a dynamical variable, on the same basis as $\vec{r}_n$.

By setting $\frac{\partial A}{\partial t_n} = 0$ we obtain

$$E_n = \frac{1}{2}\vec{v}_n^{\,2} + \frac{1}{2}\left[V(r_n) + V(r_{n-1})\right] = E_{n+1}.$$

Canonical formulation and quantization of simple discrete mechanical systems

- $\hat{L}(q, \dot{q})$ is the Lagrangian of the continuum theory.
- $t_{n+1} - t_n = \epsilon$
- generalized coordinates : t_n, q_n.
- we define the "discretized" Lagrangian

as $L(n, n+1) = L(q_n, q_{n+1}) = \epsilon \hat{L}(q, \dot{q})$

where $q = q_n$ and $\dot{q} = \frac{q_{n+1} - q_n}{\epsilon}$

- The action can be written as

$$S = \sum_{n=0}^{N} L(q_n, q_{n+1})$$

- Equations of motion

$$\frac{\partial S}{\partial q_n} = \frac{\partial L(q_{n-1}, q_n)}{\partial q_n} + \frac{\partial L(q_n, q_{n+1})}{\partial q_n} = 0$$

- Canonical conjugate momenta = ?

$$\frac{\partial L(n, n+1)}{\partial q_{n+1}} = \frac{\partial \hat{L}}{\partial \dot{q}}$$

$$\frac{\partial L(n,n+1)}{\partial q_n} = \epsilon \frac{\partial \hat{L}}{\partial q} - \frac{\partial \hat{L}}{\partial \dot{q}}$$

in the continuum case the momentum canonically conjugate to q is defined as

$$p = \frac{\partial \hat{L}}{\partial \dot{q}}$$

$$\Downarrow$$

$$p_{n+1} = \frac{\partial L(n,n+1)}{\partial q_{n+1}}$$

The Lagrange equations in the continuum case can be written as,

$$\frac{dp}{dt} = \frac{\partial \hat{L}}{\partial q}$$

Discretize the last expression!

$$\dot{p} \to \frac{p_{n+1} - p_n}{\epsilon}$$

$$\frac{p_{n+1} - p_n}{\epsilon} = \frac{\partial \hat{L}}{\partial q} = \frac{1}{\epsilon}\left[\frac{\partial \hat{L}}{\partial q} + \frac{\partial L(n,n+1)}{\partial q_n}\right]$$

$$= \frac{1}{\epsilon}\left[\frac{\partial L(n,n+1)}{\partial q_{n+1}} + \frac{\partial L(n,n+1)}{\partial q_n}\right]$$

$$= \frac{1}{\epsilon}\left[p_{n+1} + \frac{\partial L(n,n+1)}{\partial q_n}\right]$$

Then

$$p_n = -\frac{\partial L(n,n+1)}{\partial q_n}$$

Conclusion

The discret Lagrange equations are

$$\begin{cases} p_{n+1} = \dfrac{\partial L(q_n, q_{n+1})}{\partial q_{n+1}} \\ p_n = -\dfrac{\partial L(q_n, q_{n+1})}{\partial q_n} \end{cases}$$

these equations define a type 1 canonical transformation from the variables (q_n, p_n) to (q_{n+1}, p_{n+1})

The variable (q_n, p_n) constitute a phase space.

$$\{q_n, p_n\} = \{q_{n+1}, p_{n+1}\} = 1.$$

The generating function for the transformation is

$$F(q_n, p_n) = -L(q_n, q_{n+1}(q_n, p_n))$$

where $q_{n+1}(q_n, p_n)$ is given by inverting $p_{n+1} = \dfrac{\partial L(q_n, q_{n+1})}{\partial q_{n+1}}$

We have

$$\frac{\partial F}{\partial q_n} = p_n - \frac{\partial q_{n+1}}{\partial q_n} p_{n+1}$$

$$\frac{\partial F}{\partial p_n} = -\frac{\partial q_{n+1}}{\partial p_n} p_{n+1}$$

If the variables (q_n, q_{n+1}) are used for coordinates of the phase space, then one can introduce a generating function of type I

$$F_1(q_n, q_{n+1}) = - L(q_n, q_{n+1})$$

$$p_n = \frac{\partial F_1}{\partial q_{n+1}} = \{F_1, p_n\}$$

$$p_{n+1} = - \frac{\partial F_1}{\partial q_{n+1}} = -\{F_1, p_{n+1}\}$$

- if the phase space is coordinatized by (q_n, p_{n+1}) solving $p_{n+1} = \frac{\partial L(q_n, q_{n+1})}{\partial q_{n+1}}$ for q_{n+1} one can introduce a generating function of type 2

$$F_2(q_n, p_{n+1}) = p_{n+1} q_n + H(p_{n+1}, q_n)$$

such that one recovers the discrete Hamilton equations

$$q_{n+1} = q_n + \frac{\partial H}{\partial p_{n+1}}$$

$$p_n = p_{n+1} + \frac{\partial H}{\partial q_n}$$

Ex.

$$L(q_n, q_{n+1}) = m \frac{(q_{n+1} - q_n)^2}{2\epsilon} - V(q_n)\epsilon$$

$$p_{n+1} = m(q_{n+1} - q_n)/\epsilon$$

$$q_{n+1} = \frac{p_{n+1}}{\epsilon} m + q_n$$

$$F_2 = p_{n+1} q_n + \frac{p_{n+1}^2}{2m}\epsilon + V(q_{n+1})\epsilon$$

$$= p_{n+1} q_n + H(p_{n+1}, q_n)$$

$$q_{n+1} = q_n + \frac{p_{n+1}}{m}\epsilon$$

$$p_n = p_{n+1} + V'(q)\epsilon$$

Conclusion

$$q_{n+1} = q_n + \frac{p_n}{m}\epsilon - V'(q_n)\frac{\epsilon^2}{m}$$

$$p_{n+1} = p_n - V'(q_n)\,\epsilon$$

Quantization

- We choose a polarization such that the wavefunctions are functions of the configuration variables $\psi(q_{n+1})$.
- The evolution of the system is implemented via a unitary transformation.
- $p_{n+1} = U p_n U^+$

  $q_{n+1} = U q_n U^+$

  $U = \exp\left\{ i\frac{V(q_n)\epsilon}{\hbar}\right\} \exp\left\{ i\frac{p_n^2 \epsilon}{2m\hbar}\right\}$

At a quantum mechanical level the energy $H^0(q_{n+1}, p_{n+1})$

is not conserved!

One construct an "energy" (both at a quantum mechanical and classical level) that is conserved by the discrete evolution

$$\exp(X)\exp(Y) = \exp\Big(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}\big([X,[X,Y]],[Y,[Y,X]]\big) + \dots\Big)$$

and one can therefore write

$$U = \exp\Big(\frac{i\epsilon}{\hbar} H_{eff}(q_i, p_i)\Big)$$

where $H_{eff} = H^0(q, p) + O(\epsilon^2)$ is immediately conserved under the evolution!

Discrete quantum gravity: applications to cosmology

- Consider a constrained mechanical system.
- we replace in the action all time derivatives with discrete first order derivatives
- The time integral in the action is replaced by the sum

$$S = \sum_{n=1}^{N} L(q_n, q_{n+1}) \text{ with}$$

$$L(n, n+1) = p_n (q_{n+1} - q_n) - \varepsilon H(q_n, p_n) - \lambda_{nB} \phi^B (q_n, p_n).$$

- $t_{n+1} - t_n = \varepsilon$
- we assume a Hamiltonian H and M constraints $B = 1, \dots M$.

<u>Remark</u>

in a theory where time is discretized it make no sense to work with a Hamiltonian since it is the generator of infinitesimal time evolution and one cannot have infinitesimal evolution if the time is discrete.!

- The time evolution should be described via a canonical transformation that implements the discrete time evolution between instants n and $n+1$.

Example

A type I of a canonical transformation that accomplishes this task has as generating function minus the Lagrangian viewed as a function of q_n and q_{n+1}. From it, one obtains at instant $n+1$,

$$P^{q}_{n+1} = \frac{\partial L(n,n+1)}{\partial q_{n+1}} = p_n$$

$$P^{p}_{n+1} = \frac{\partial L(n,n+1)}{\partial p_{n+1}} = 0$$

$$p^{\lambda B}_{n+1} = \frac{\partial L(n,n+1)}{\partial \lambda_{(n+1)B}} = 0$$

Similarly, the momenta at instant n are given by

$$P_n^q = -\frac{\partial L(n,n+1)}{\partial q_n} = P_n + \epsilon \frac{\partial H(q_n,p_n)}{\partial q_n} + \lambda_{nB} \frac{\partial \phi^B(q_n,p_n)}{\partial q_n}$$

$$P_n^p = -\frac{\partial L(n,n+1)}{\partial p_n} = -(q_{n+1}-q_n) + \varepsilon \frac{\partial H(q_n,p_n)}{\partial p_n} + \lambda_{nB} \frac{\partial \phi^B}{\partial p_n}(q_n,p_n)$$

$$P_n^{\lambda_B} = \phi^B(q_n,p_n)$$

Combining these two pairs of equations we obtain

$$p_n - p_{n-1} = -\varepsilon \frac{\partial H}{\partial q_n}(q_n,p_n) - \lambda_{nB} \frac{\partial \phi^B}{\partial q_n}(q_n,p_n)$$

$$q_{n+1} - q_n = \varepsilon \frac{\partial H(q_n,p_n)}{\partial p_n} + \lambda_{nB} \frac{\partial \phi^B}{\partial p_n}(q_n,p_n)$$

$$\phi^B(q_n,p_n) = 0$$

Remark:

The above equations appear entirely equivalent to the continuum ones. However, they hide the fact that in order for the constraints to be preserved, the Lagrange multipliers get fixed.

Therefore, it is illuminating to rewrite these equations in terms of the canonically conjugate pair,

$$p^q_{n+1} - p^q_n = -\varepsilon \frac{\partial H(q_n, p^q_{n+1})}{\partial q_n} - \lambda_{nB} \frac{\partial \phi^B(q_n, p^q_{n+1})}{\partial q_n}$$

$$q_{n+1} - q_n = \varepsilon \frac{\partial H(q_n, p^q_{n+1})}{\partial p^q_{n+1}} + \lambda_{nB} \frac{\partial \phi^B(q_n, p^q_{n+1})}{\partial p^q_{n+1}}$$

$$\phi^B(q_n, p^q_{n+1}) = 0$$

Questions about this construction

i) Solubility of the multiplier equations
ii) Performing meaningful comparisons
iii) The continuum limit
iv) Singularities
v) Problem of time
vi) Discretization ambiguities

Part II

Fractional Calculus and Variational Principles

History:

1695, L'Hospital: "what is n fractional"?

Leibniz: "This is an apparent paradox from which, one day, useful consequences will be drawn".

1819 - S.F. Lacroix, two pages about a derivative of arbitrary order

1822 - J.B.J. Fourier

1823 - Abel

1832-1837 Liouville

$\boxed{1974}$

The fractional calculus has been developed after 1974 intensively!!

FRACTIONAL CALCULUS

- Materials Modeling I
- Physical and Biological Systems Modeling
- Control
- Anomalous transport II
- Physics
- Mathematics
- Finance

⋮

REFERENCES

1) K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York - London, 1974.

2) I. Podlubny, Fractional Differential Equations, Academic Press, 1999.

3) K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons Inc. New York, 1993

4) R. Hilfer (Ed.) Applications of Fractional calculus in physics, Word Scientific Publishing company, Singapore, New Jersey, London and Hong Kong (2000)

5) S. G. Samko, A. K. Kilbas, O. I. Marichev, Fractional integrals and Derivatives Theory and Applications, Gordon and Breach Science Publishers, (1993).

6) The Fractional Fourier Transform with applications in Optics and Signal Processing
H. M. Ozaktas, Z. Zalevsky, M. A. Kutay, Wiley & Sons, 2001

7) V. Kiryakova, "Generalized Fractional calculus and applications" Pitman Research Notes in Mathematics No. 301. Longman Scientific & Technical, Harlow 1994.

1. S. Muslih and D. Baleanu, in press, J. Math. Anal. Appl. (2005)

2. D. Baleanu and T. Avkar, Nuovo Cimento B 119, 73 (2004)

3. D. Baleanu and S. Muslih, accepted for publication in Physica Scripta (2005).

4. S. Muslih and D. Baleanu, accepted for publication in Czech. J. Phys. (2005)

5. D. Baleanu, "Constrained systems and Riemann-Liouville fractional derivative", Proceedings of 1st IFAC workshop on Fractional Differentiation and its Applications, Bordeaux, France, July 19-21, (2004) p. 597

Gamma Function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt$$

$$\Gamma(z+1) = z\,\Gamma(z)$$

$$\Gamma(n+1) = n!$$

Limit Representation

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, z^n}{z(z+1)\ldots(z+n)}$$

Beta Function

$$B(z,w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1}\, d\tau$$

$\mathrm{Re}(z) > 0$, $\mathrm{Re}(w) > 0$

$$B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}$$

Riemann-Liouville fractional derivatives

The left Riemann-Liouville fractional derivative is defined as

$$
{}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau
$$

The right Riemann-Liouville fractional derivative has the form

$$
{}_tD_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_\tau^b (\tau-t)^{n-\alpha-1} f(\tau)\,d\tau
$$

$n-1 \leq \alpha < n$

- If α is an integer we obtain

$$
{}_aD_t^{\alpha} f(t) = \left(\frac{d}{dt}\right)^{\alpha}
$$

$$
{}_tD_b^{\alpha} f(t) = \left(-\frac{d}{dt}\right)^{\alpha}, \quad \alpha = 1, 2, \ldots
$$

Under the assumptions that $f(t)$ is continuous and $p \geq q \geq 0$, the most general property of RL fractional derivatives can be written as

$$ {}_aD_t^{p} \left({}_aD_t^{-q} f(t)\right) = {}_aD_t^{p-q} f(t) $$

. For $p > 0$ and $t > a$ we obtain

$$ {}_aD_t^{p} \left({}_aD_t^{-p} f(t)\right) = f(t) \quad (*) $$

$*$ - is called the fundamental property of the RL fractional derivative.

Ex. $\; {}_aD_t^{p} (t-a)^{\nu} = \dfrac{\Gamma(\nu+1)}{\Gamma(-p+\nu+1)} (t-a)^{\nu-p}$

$\nu > -1$

∴ the fractional RL derivative of a constant is not zero.

- $\frac{d^m}{dt^m}$ and ${}_aD_t^p$ commute only if
$f^{(j)}(a) = 0$, $j = 0, 1, \dots m-1$ is fulfilled

- ${}_aD_t^p$ and ${}_aD_t^q$ commute only if

$$\left[{}_aD_t^{p-j} f(t)\right]_{t=a} = 0, \quad j = 1, \dots m$$

and

$$\left[{}_aD_t^{q-j}\right]_{t=a} = 0, \quad j = 1, \dots, m$$

Remark:
Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $f(a)$, $f'(a)$, ...
RL approach leads to initial conditions containing the limit values of RL derivatives at $t = a$:

$\lim_{t \to a} {}_aD_t^{\alpha-1} f(t) = b_1$, $\lim_{t \to a} {}_aD_t^{\alpha-2} f(t) = b_2, \dots$

$\lim_{t \to a} {}_aD_t^{\alpha-n} f(t) = b_n$. , b_k, $k = 1, 2, \dots n$ are given constants.

?

Caputo's Fractional Derivative

[M. Caputo, Linear model of dissipation whose Q is almost frequency independent -II, Geophys. J R. Astr. Soc. vol 13, 1967, pp. 529-539.]

$$ {}^{c}_{a}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_{a}^{t} \frac{f^{(n)}(\tau)\, d\tau}{(t-\tau)^{\alpha+1-n}} $$

$n-1 < \alpha < n$.

. $\alpha \to n \Rightarrow \lim_{\alpha \to n} {}^{c}_{a}D^{\alpha}_{t} f(t) = f^{(n)}(t)$

$n = 1, 2 \ldots$

. Advantages:

A) $\lim_{\alpha \to n} {}^{c}_{a}D^{\alpha}_{t} f(t) = f^{(n)}(a) + \int_{a}^{t} f^{(n+1)}(\tau)\, d\tau$, $n = 1, 2 \ldots$

$\Downarrow$

Caputo derivative take on the same form as for integer-order differential equations !!

B) RL: ${}_0D_t^\alpha \mathcal{D} = \dfrac{\mathcal{D}\, t^{-\alpha}}{\Gamma(1-\alpha)} \neq 0$, $\mathcal{D} = \text{const}$

${}^c_0D_t^\alpha \mathcal{D} = 0$!!

C) Laplace transform of RL fractional derivative is:

$$\int_0^\infty e^{-pt} \{ {}_0D_t^\alpha f(t) \} dt = p^\alpha F(p) - \sum_{k=0}^{n-1} p^k \, {}_0D_t^{\alpha-k-1} f(t) \Big|_{t=0}$$

$(n-1 \leq \alpha < n)$.

Laplace transform of the Caputo derivative is

$$\int_0^\infty e^{-pt} \{ {}^c_0D_t^\alpha f(t) \} dt = p^\alpha F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} f^{(k)}(0).$$

$(n-1 < \alpha \leq n)$

Fractional Lagrangian and Hamiltnian mechanics

- F. Riewe, Phys. Rev. E, 53(2), 1996, p. 1890
- F. Riewe, Phys. Rev. E, 55(3), 1997, p. 3581.

!! There is no direct method of applying variational principle to nonconservative systems, which are characterized by friction or other dissipative processes. (P. S. Bauer, Proc. Natl. Acad. Sci. USA 17, 1931, p. 311)

A Solution is:
Use the fractional calculus!!

Hint:

Use the theories with Higher Derivatives

Let $L(q, \dot{q}, \ddot{q}, \dots)$ be a theory with higher derivatives of orders $\{N_a\}$.

in this case, the Lagrange equations that follow extremality of the action are of the form

$$\frac{\delta S}{\delta q^a} = \sum_{l=0}^{N_a} (-1)^l \frac{d^l}{dt^l} \left(\frac{\partial L}{\partial q^{a(l)}} \right) = 0$$

Riewe's formulation

Let us consider the action function of the form

$$S = \int_a^b L(\{q_n^r, Q_{n'}^r\}, t)\, dt$$

where the generalized coordinates are defined as follows

$$q_n^r = \left({}_aD_t^\alpha\right)^n x_r(t)$$

$$Q_{n'}^r = \left({}_tD_b^\alpha\right)^{n'} x_r(t)$$

$r = 1, 2, \dots R$, denotes the number of fundamental coordinates

$n = 0, \dots, N$, the sequential order of the derivatives defining the generalized coordinates q and

$n' = 1, \dots N'$ the sequential order

of the derivatives in definition of the coordinates Q.

Euler-Lagrange equations:

$$\frac{\partial L}{\partial q_0^r} + \sum_{n=1}^{N}\left({}_tD_b^\alpha\right)^n \frac{\partial L}{\partial q_n^r} + \sum_{n'=1}^{N'}\left({}_aD_t^\alpha\right)^{n'} \frac{\partial L}{\partial Q_{n'}^r} = 0$$

The generalized momenta take the form

$$p_n^r = \sum_{k=n+1}^{N}\left({}_tD_b^\alpha\right)^{k-n-1} \frac{\partial L}{\partial q_k^r}$$

$$\pi_{n'}^r = \sum_{k=n'+1}^{N'}\left({}_aD_t^\alpha\right)^{k-n'-1} \frac{\partial L}{\partial Q_k^r}$$

The canonical Hamiltonian has the form:

$$H = \sum_{r=1}^{R}\sum_{n=0}^{N-1} p_n^r q_{n+1}^r + \sum_{r=1}^{R}\sum_{n'=0}^{N'-1} \pi_{n'}^r Q_{n'+1}^r - L$$

Example

$$L = \frac{1}{2} m \dot{x}^2 - V(x) + \frac{i}{2} \gamma x^2_{(\frac{1}{2}, b)}$$

or

$$L = \frac{1}{2} m q_1^2 - V(q_0) + i \frac{1}{2} \gamma q^2_{\frac{1}{2}}$$

Euler-Lagrange equations is

$$\frac{\partial L}{\partial q_0} + i \frac{d^{\frac{1}{2}}}{d(t-b)^{\frac{1}{2}}} \frac{\partial L}{\partial q_{\frac{1}{2}}} - \frac{d}{dt} \frac{\partial L}{\partial q_1} = 0$$

$$\Downarrow$$

$$m \ddot{x} = - \gamma \dot{x} - \frac{\partial V}{\partial x}$$

Using the fact that

$$\underline{{}_bD_t^{\frac{1}{2}} \; {}_bD_t^{\frac{1}{2}} x = {}_bD_t x} = \dot{x}$$

we obtain the momenta as

$$P_0 = \frac{\partial L}{\partial q_{\frac{1}{2}}} + i \, {}_bD_t^{\frac{1}{2}} \left(\frac{\partial L}{\partial q_1} \right) = i \gamma x_{(\frac{1}{2}, b)} + i m x_{(\frac{3}{2}, b)}$$

$$p_{\frac{1}{2}} = \frac{\partial L}{\partial q_1} = m\dot{x}$$

The Hamiltonian is

$$H = q_{\frac{1}{2}} p_0 + q_1 p_{\frac{1}{2}} - L$$

$$= \frac{p_{\frac{1}{2}}^2}{2m} + q_{\frac{1}{2}} p_0 + V - i\frac{1}{2}\gamma q_{\frac{1}{2}}^2$$

Hamilton's equations are

$$\frac{\partial H}{\partial q_0} = i\, {}_bD_t^{\frac{1}{2}} p_0, \quad \frac{\partial H}{\partial p_0} = q_{\frac{1}{2}}$$

$$\frac{\partial H}{\partial q_{\frac{1}{2}}} = i\, {}_bD_t^{\frac{1}{2}} p_{\frac{1}{2}}, \quad \frac{\partial H}{\partial p_{\frac{1}{2}}} = q_1.$$

Extension of Riewe's fractional Hamiltonian formulation

Let us consider the following Lagrangian:

$$\bar{L}\left(\{q^r_n, Q^r_{n'}\}, t, \lambda_m(t)\right) =$$

$$L\left(\{q^r_n, Q^r_{n'}\}, t\right) + \lambda_m^{(t)} \phi_m\left(t, q_0^1, \ldots q_0^R, q_n^r, Q_{n'}^r\right).$$

The left and the right canonical momenta are defined as follows:

$$p_n^r = \sum_{k=n+1}^{N} \left({}_tD_b^\alpha\right)^{k-n-1} \frac{\partial \bar{L}}{\partial q_k^r}$$

$$\pi_{n'}^r = \sum_{k=n'+1}^{N'} \left({}_aD_t^\alpha\right)^{k-n'-1} \frac{\partial \bar{L}}{\partial Q_k^r}$$

The canonical Hamiltonian has the following form:

$$\bar{H} = \sum_{r=1}^{R} \sum_{n=0}^{N-1} p_n^r q_{n+1}^r + \sum_{r=1}^{R} \sum_{n'=0}^{N'-1} \pi_{n'}^r Q_{n'+1}^r - \bar{L}$$

$$\{q_n^r, \bar{H}\} = {}_tD_b^\alpha p_n^r$$

$$\{Q_{n'}^r, \bar{H}\} = {}_aD_t^\alpha \pi_{n'}^r$$

$$\{q_0^r, \bar{H}\} = {}_tD_b^\alpha p_0^r + {}_aD_t^\alpha \pi_0^r$$

$$n = 1, \dots N, \quad n' = 1, \dots, N'$$

$$\{p_n^r, \bar{H}\} = q_{n+1}^r = {}_aD_t^\alpha q_n^r$$

$$\{\pi_{n'}^r, \bar{H}\} = Q_{n+1}^r = {}_tD_b^\alpha Q_{n'}^r$$

$$\frac{\partial \bar{H}}{\partial t} = -\frac{\partial \bar{L}}{\partial t}$$

Ex.

$$S[x_1, x_2] = \frac{1}{2}\int_0^1 [x_1^2 + x_2^2]\,dt$$

such that

$$ {}_0D_t^\alpha x_1 = -x_1 + x_2$$

$$x_1(0) = 1$$

The modified Lagrangian $\bar{L}$ is given by

$$\bar{L} = \frac{1}{2}[x_1^2 + x_2^2] + l\phi_1 + \lambda\phi_2,$$

where $\phi_1 = {}_0D_t^\alpha x_1 + x_1 - x_2$

$\phi_2 = x_1(0) - 1 = 0$

The generalized canonical momenta are defined as

$$p_0^1 = l, \quad p_0^2 = 0$$

The Hamiltonian $\bar{H}$ is given by

$$\bar{H} = p_0^1 q_1^1 - \frac{1}{2}[x_1^2 + x_2^2] - l\phi_1 - \lambda\phi_2$$

Equivalence of fractional Hamiltonian and Lagrangian formulations for systems with linear velocities

$$L' = a_j(q^i)\,{}_aD_t^\alpha q^j - V(q^i) .$$

Let us define

$$x_j^n = \left({}_aD_t^\alpha\right)^n q_j \,, \quad n = 0, 1, \dots N-1$$
$$j = 1, 2, \dots R$$

The generalized momenta are given by

$$P_j^0 = \frac{\partial L}{\partial x_j^0} = a_j(x_i^0) \,, \quad P_j^1 = \frac{\partial L}{\partial x_j^1} = 0$$

The canonical Hamiltonian reads as

$$H = \left(P_j^0 - a_j(x_i^0)\right) x_j^1 + V(x_i^0)$$

The Hamiltonian equations of motion are calculated as

$$(1)\quad \frac{\partial H}{\partial x_j^1} = p_j^0 - a_j(x_i^0) = ({}_tD_b^\alpha)\, p_j^1 = 0$$

The other equations of motion are calculated as:

$$(2)\quad \frac{\partial H}{\partial x_k^0} = -\frac{\partial a_j(x_i^0)}{\partial x_k^0} x_j^1 + \frac{\partial V(x_i^0)}{\partial q^k}$$

$$= ({}_tD_b^\alpha)\, p_k^0$$

$$(3)\quad \frac{\partial H}{\partial p_k^0} = x_k^1 = ({}_aD_t^\alpha)\, q_k$$

From (1), (2) and (3) we obtain

$$\frac{\partial a_j(q^i)}{\partial q^k}\, {}_aD_t^\alpha q^j + {}_tD_b^\alpha a_k(q^i) - \frac{\partial V(q^i)}{\partial q^k} = 0$$

"

E-L equations!!

Fractional Euler-Lagrange equations for Lagrangians with linear velocities

- Agrawal P.O., Nonlinear Dynamics vol 29, p.145, (2002).
- D. Baleanu, T. Avkar, Nuovo Cimento B, vol. 119, no. 1, p. 73, (2004).

Let $J[q^1, \dots, q^n]$ be a functional of the form

$$\int_a^b L(t, q^1, \dots, q^n, {}_aD_t^\alpha q^1, \dots, {}_aD_t^\alpha q^n, {}_tD_b^\beta q^1, \dots, {}_tD_b^\beta q^n)\,dt$$

defined on the set of functions $q^i(t)$, $i = 1, \dots, n$ which have continuous left RL fractional derivative of order α and right RL fractional derivative of order β in $[a,b]$ and satisfy the boundary conditions $q^i(a) = q^i_a$, $q^i_b = q^i_b$.

A necessary condition for $J[q^1, \dots, q^n]$ to admit an extremum for given functions $q^i(t)$, $i = 1, \dots, n$ is that $q^i(t)$ satisfy Euler-Lagrange equations

$$\frac{\partial L}{\partial q^j} + {}_tD_b^{\alpha} \frac{\partial L}{\partial {}_aD_t^{\alpha} q^j} + {}_aD_t^{\beta} \frac{\partial L}{\partial {}_tD_b^{\beta} q^j} = 0$$

$j = 1, \dots, n.$

<u>Ex.</u>

$$L = a_j(q^i)\, q^j - V(q^i), \quad (1)$$

$a_j(q^i)$ and $V(q^i)$ are functions of their arguments.

Aim: Construct the corresponding fractional generalization of the Lagrangian (1)!!

$0 < \alpha \leq 1$

A)

$$L' = a_j(q^i)\, {}_aD_t^\alpha q^j - V(q^i)$$

$$\Downarrow$$

$$\frac{\partial a_j(q^i)}{\partial q^k}\, {}_aD_t^\alpha q^j + {}_tD_b^\alpha a_k(q^i) - \frac{\partial V(q^i)}{\partial q^k} = 0$$

B)

$$L'' = -a_j(q^i)\, {}_tD_b^\alpha q^j - V(q^i)$$

$$\frac{\partial a_j(q^i)}{\partial q^k}\, {}_tD_b^\alpha q^i + {}_aD_t^\alpha a_k(q^i) + \frac{\partial V(q^i)}{\partial q^k} = 0$$

Ex.

$$L = \dot q^1 q^2 + \dot q^3 q^4 - V(q^2, q^3, q^4),$$

$$V(q^2, q^3, q^4) = -\frac{1}{2}\left[(q^4)^2 - 2q^2 q^3\right].$$

The above Lagrangian is a second-class constrained system in Dirac classification

$$L' = -\left[\left({}_tD_b^\alpha q^1\right) q^2 + \left({}_tD_b^\alpha q^3\right) q^4 + V(q^2, q^3, q^4) \right]$$

Euler-Lagrange equations are:

$$
\begin{aligned}
&{}_aD_t^\alpha q^2 = 0 \\
&{}_tD_b^\alpha q^1 + q^3 = 0 \qquad\qquad (1)\\
&{}_aD_t^\alpha q^4 + q^2 = 0 \\
&{}_tD_b^\alpha q^3 - q^4 = 0
\end{aligned}
$$

A solution of (1) is:

$$q^2(t) = C_1 (t-a)^{\alpha-1}$$

$$q^4(t) = C_2 (t-a)^{\alpha-1} - \frac{C_1}{\Gamma(\alpha)} \int_t^b (-t+z)^{\alpha-1} (y-a)^{\alpha-1}\, dy\, dz$$

$$q^3(t) = C_3(b-t)^{\alpha-1} + \frac{C_2}{\Gamma(\alpha)}\int_t^b (-t+z)^{\alpha-1}(-a+z)\,dz$$

$$-\frac{C_1}{\Gamma(\alpha)^2}\int_t^b (-t+z)^{\alpha-1}\int_a^z (z-y)^{\alpha-1}(y-a)^{\alpha-1}\,dy\,dz$$

$$q^1(t) = C_4(b-t)^{\alpha-1} - \frac{C_3}{\Gamma(\alpha)}\int_t^b (-t+z)^{\alpha-1}(b-z)^{\alpha-1}\,dz$$

$$-\frac{C_2}{\Gamma(\alpha)^2}\int_t^b (-t+z)^{\alpha-1}\int_z^b (-z+y)^{\alpha-1}(-a+y)^{\alpha-1}\,dy\,dz$$

$$+\frac{C_1}{\Gamma(\alpha)^3}\int_t^b (\tau-t)^{\alpha-1}\int_\tau^b (\tau-y)^{\alpha-1}\int_a^y (z-y)^{\alpha-1}(y-a)^{\alpha-1}\,dz\,dy\,d\tau$$

C_1, C_2, C_3, C_4 are constants.

If $\alpha \to 1$, $a \to 0$, $b \to 1$ we recover

$$q^1(t) = \frac{C_4' t^3}{6} - \frac{C_3' t^2}{2} + C_2' t + C_1'$$

$$q^2(t) = C_4', \quad q^3(t) = \frac{C_4'}{2}t^2 - C_3' t + C_2'$$

$$q^4(t) = -C_4' t + C_3'.$$

Open Problems

1) Fractional stochastic quantization of constrained systems

2) Fractional canonical quantization of constrained systems

3) Softwares for systems of fractional differential equations