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Non-commutative Geometry and Gravity

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Motivations

- Spacetime at Planck lenght (10^{-33}cm) has not to be a continuum of points a discretized, cell-like or noncommutative structure can emerge

- General relativity + Quantum mechanics $\downarrow$ Quantum gravity — Geometry + Noncommutative Observables $\downarrow$ Noncommutative Geometry

General relativity	Geometry
+	+
Quantum mechanics	Noncommutative Observables
$\downarrow$	$\downarrow$
Quantum gravity	Noncommutative Geometry

- Can Riemannian geometry be consistently deformed in Noncommutative Riemmanian geometry?

- Is noncommutative gravity quantizable?
  Noncommutativity helps rinormabizability?
  Noncommutativity is a good regularization tool?

Noncommutative Diffeomorphisms approach:

given a space with a noncommutative structure, we deform the algebra of infinitesimal diffeomorphisms (coord. transf.) so that the noncommutative space has a symmetry structure as big as the corresponding commutative case.

For a different approach, see [Madore] [Buric, Madore]

If instead of the ∞-dimensional Lie algebra of infinitesimal diffeomorphisms we consider that of a Lie group we obtain quantum groups and quantum spaces.

The $\theta^{\mu\nu}$-constant Minkowski space that is invariant under $\theta^{\mu\nu}$-Poincaré group is an example [Dimitrijević, Wess] [Wess] [Chaichian, Kulish Nishijima, Tureanu] [Koch, Tsouchnika]

$$\begin{bmatrix}\text{Manifold}\\ M\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Vectorfields}\\ \Xi\end{bmatrix} \longrightarrow \begin{bmatrix}\text{1-forms}\\ \Lambda\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Diff. calculus}\\ d\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Connections}\\ \nabla\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Metric } g_{\mu\nu}\\ \text{Riemannian}\\ \text{Geometry}\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Curvature } R\\ \text{Eintein's}\\ \text{equations}\end{bmatrix}$$

$$\begin{bmatrix}\text{Noncomm.}\\ \text{Algebra}\\ A_*\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Vectorfields}\\ \Xi_*\end{bmatrix} \longrightarrow \begin{bmatrix}\text{1-forms}\\ \Lambda_*\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Diff. calculus}\\ d\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Connections}\\ \nabla\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Metric } g_{*\mu\nu}\\ \text{Riemannian}\\ \text{Geometry}_*\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Curvature } R_*\\ \text{Einstein's}\\ \text{equations}_*\end{bmatrix}$$

$\|$

$$\left[\begin{array}{l}\text{Infinitesimal diffeomorphisms (coord. transformations)}\\ \text{They form a Lie algebra, it is useful to consider}\\ \mathcal{U}\Xi = \text{Universal enveloping algebra of } \Xi\end{array}\right]$$

[illegible]

$\|$

$$\left[\begin{array}{l}\text{Infinitesimal noncommutative diffeomorphisms,}\\ \text{they form a deformed Lie algebra}\\ \mathcal{U}\Xi_\star = \text{Universal enveloping algebra of } \Xi_\star\end{array}\right]$$

$\mathcal{U}\Xi \simeq \mathcal{U}\Xi_\star$ as algebras, but not as Hopf algebras.

1). • This program has been carried out in the θ-constant case.

2). • We are studying the general $\theta^{\mu\nu}(x)$ - case:

$$[x^\mu, x^\nu] = i\hbar \theta^{\mu\nu}(x)$$

One approach is to use Konstevich $*$-product

In this perspective one studies gravity locally (on opens of $\mathbb{R}^4$ with coordinates x^μ) and then the knowledge of noncommutative diffeomorphisms should provide the global picture (going from chart to chart).

3). • We here present and carry out the noncommutative gravity program in the case of $*$-products coming from twists.

- Quite general class of $*$-products
- Global aspects can be studied

Star products from Twists $\mathcal{F}$

Ex 1 $$(h * g)(x) = e^{\frac{i}{2}\hbar\,\theta^{\mu\nu}\partial_\mu \otimes \partial'_\nu}\; h(x)\, g(x')\Big|_{x' \to x}$$

$$\mathcal{F} = e^{-\frac{i}{2}\hbar\,\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$$ is a twist ($\theta^{\mu\nu}$ = constants)

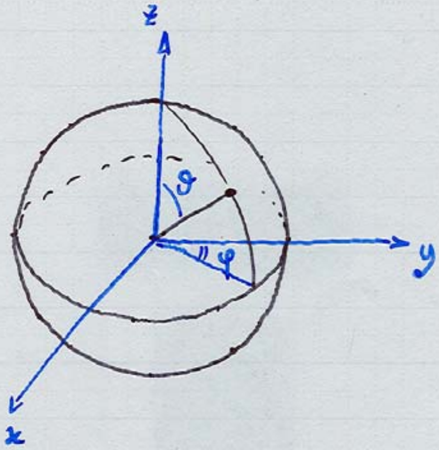
General Definition

- $\mathcal{F} \in U\Xi \otimes U\Xi$
- $\mathcal{F}$ is invertible, $\mathcal{F}^{-1} = e^{+\frac{i}{2}\hbar\,\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$
- $(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F}$ in $U\Xi \otimes U\Xi \otimes U\Xi$
- $(\varepsilon \otimes id)\mathcal{F} = 1 = (id \otimes \varepsilon)\mathcal{F}$ (normalization condition)

Ex 2 The q-plane $xy = q\,yx$ can be obtained from the twist

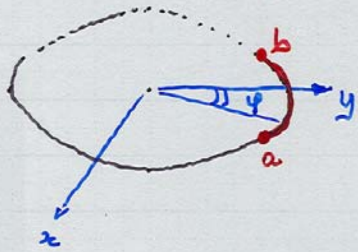
$$\mathcal{F} = e^{ih\, x\frac{\partial}{\partial x} \otimes y\frac{\partial}{\partial y}} \qquad (q = e^{ih})$$

Ex 3

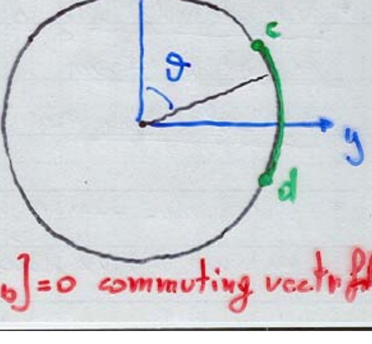


$$\mathcal{F} = e^{i\hbar f(\varphi)\partial_\varphi \otimes h(\vartheta)\partial_\vartheta}$$

with $f(\varphi)$ zero outside the interval a — b



and $h(\vartheta)$ zero outside the interval c — d



In general $\mathcal{F} = e^{i\hbar\sigma^{ab} X_a \otimes X_b}$; $[X_a, X_b] = 0$ commuting vect. fields.

Definition

A noncommutative manifold is a couple $(M, \mathcal{F})$ where M is a smooth commutative manifold and $\mathcal{F} \in U\Xi \otimes U\Xi$ is a twist.

On the algebra of functions on the manifold M we have then the following $*$-product:

$$h * g = \bar{f}^{\alpha}(h) \cdot \bar{f}_{\alpha}(g)$$

where $\mathcal{F}^{-1} = \bar{f}^{\alpha} \otimes \bar{f}_{\alpha} \in U\Xi \otimes U\Xi$

$\mathcal{F} = f^{\alpha} \otimes f_{\alpha} \in U\Xi \otimes U\Xi$

$$\left(\text{e.g. } \mathcal{F} = e^{i\hbar\vartheta\partial\otimes\partial} = 1 \otimes 1 + i\hbar\vartheta\partial \otimes \partial - \tfrac{1}{2}\hbar^2\vartheta\vartheta\partial\partial \otimes \partial\partial + \ldots \right)$$

$\bar{f}^{\alpha} \in U\Xi$; $\bar{f}^{\alpha} = \ldots + v'v'' + \ldots$; $v'v''(h) = v'(v''(h))$

$\bar{\mathcal{F}}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha$ not only acts on functions;

it also acts on vectorfields

Ξ_* is deformed Lie algebra of vectorfields:

$$[u, v]_* := [\bar{f}^\alpha(u), \bar{f}_\alpha(v)] \qquad v'v''(u) = v'(v''(u))$$

$\Xi = \Xi_*$

$\uparrow$ as vectorspaces; but $[\;]_* \neq [\;]$.

Classically $\qquad \mathcal{L}_u v = [u, v]$

then we have a $*$-Lie derivative

$$\mathcal{L}^*_u v = [u, v]_*$$

Also on functions:

$$\mathcal{L}^*_u(h) = \mathcal{L}_{\bar{f}^\alpha(u)} \bar{f}_\alpha(h) = \mathcal{L}_{\bar{f}^\alpha(u)\bar{f}_\alpha}(h)$$

Leibnitz rule of $\mathcal{L}^*_u$:

$$(1) \qquad \mathcal{L}^*_u(h*g) = \mathcal{L}^*_u(h)*g + \bar{R}^\alpha(h)*\mathcal{L}^*_{\bar{R}_\alpha(u)}(g)$$

[Woronowicz]

$$R = \mathcal{F}_{21}\mathcal{F}^{-1} \in U\Xi \otimes U\Xi$$

$$R^{-1} = \mathcal{F}\mathcal{F}_{21}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha$$

There is a coproduct

$$\Delta_*$$

that implements (1).

Ξ Lie alg. of vectorfields

is also an A-module

$v \in \Xi \qquad hv \in \Xi$

$\Xi_\star$ $\star$-Lie alg. of vectorfields

is a $A_\star$-module [Majid]

$v \in \Xi_\star \qquad h \star v \in \Xi_\star$

- $$h \star v = \bar{f}^\alpha(h)\, \bar{f}_\alpha(v)$$

- $$\mathcal{L}^\star_{g \star u}(h) = g \star \mathcal{L}^\star_u(h)$$

$A_\star$ - linearity of $\mathcal{L}^\star$ on functions.

$\boxed{+\ \text{twist}\ \mathcal{F}}$

$$\downarrow \mathcal{F} \qquad \begin{array}{c} \mathcal{U}\Xi \\ \downarrow \mathcal{F} \\ \mathcal{U}\Xi_* \end{array}$$

Space of 1-forms Λ ($A_\star$-linear functionals on $\Xi_\star$)

$$\langle v, \omega \rangle_\star \in A_\star$$

$$\langle h \star v, \omega \rangle_\star = h \star \langle v, \omega \rangle_\star$$

Λ is also an $A_\star$-module:

$$\langle v, h \star \omega \rangle_\star := \bar{R}^\alpha(h) \star \langle \bar{R}_\alpha(v), \omega \rangle_\star$$

Λ is actually a $A_\star$-bimodule:

$$\omega \star h := \bar{R}^\alpha(h) \star \bar{R}_\alpha(\omega)$$

Exterior differential

$d_\star h$ such that $\langle v, d_\star h \rangle_\star = \mathcal{L}^\star_v h \quad \forall v$

If $\langle E_i, \omega^j \rangle_\star = \delta_i^j$ then $d_\star h = \omega^i \star \mathcal{L}^\star_{E_i} h$.

Thm $d_* = d$

Covariant derivative - Connection.

$$\nabla^*_v : \Xi_* \longrightarrow \Xi_*$$

- $$\nabla^*_{h*v+w} u = h * \nabla^*_v u + \nabla^*_w u$$

- $$\nabla^*_v (\lambda u + u') = \lambda \nabla^*_v u + \nabla^*_v u'$$

- $$\nabla^*_v (h * u) = \mathcal{L}^*_v (h) * u + \bar{R}^\alpha(h) * \nabla^*_{\bar{R}_\alpha(v)} u$$

- $$\nabla^* : \Xi_* \longrightarrow \Lambda_* \otimes_* \Xi_*$$

- $$\nabla^*(h * u) = dh \otimes_* u + h * \nabla^*(u)$$

1) then $\nabla^*_v u = \langle v, \nabla^* u \rangle_*$

2) **Thm** $\nabla_* = \nabla$

Curvature and Torsion

$$T^*(u,v) := \nabla^*_u v - \nabla^*_{\bar{R}^*(v)} \bar{R}_\alpha(u) - [u,v]_*$$

$$R^*(u,v)w := \nabla^*_u \nabla^*_v w - \nabla^*_{\bar{R}^*(v)} \nabla^*_{\bar{R}_\alpha(u)} w - \nabla^*_{[u,v]_*} w$$

T^* and R^* are A_*-linear (well defined!)

$$T^*(h*u, v) = h * T^*(u,v)$$

$$R^*(h*u, v)w = h * R^*(u,v)w$$

If $\{\omega^i\}$ are a basis of 1-forms

$$R^* = \frac{1}{2}\, \omega^k \otimes_* (\omega^i \wedge_* \omega^j) * R^{*\ell}{}_{kij} * E_\ell$$

Ricci tensor

$$\mathrm{Ric}(u,w) = \left\langle R^*(u,E_i)w,\ \omega^i \right\rangle_*$$

Metric

- $$\mathfrak{g} \in \Lambda \otimes_* \Lambda$$

$$\mathfrak{g} = \omega^i * g_{ij} \otimes_* \omega^j$$

- $*$- Symmetric $\quad \bar{R}^\alpha(\omega^j) \otimes_* \bar{R}_\alpha(\omega^i * g_{ij}) = \mathfrak{g}$

- Hermitian $\quad \mathfrak{g}^\dagger = \mathfrak{g}$

- Nondegenerate : g_{ij} invertible i.e.

$$G^{ki} * g_{ij} = \delta^k_j$$

Thm (Fund. theorem)

There exists a unique torsion-free and metric-compatible connection.

Given the ~~*~~inverse metric G^{ij}

$$G^{ij} * g_{jk} = \delta^i_k$$

$$g_{ij} * G^{jk} = \delta_i^k$$

We construct the bi-vector

$$\mathcal{G} = E_i \otimes_* G^{ij} * E_j$$

inverse of the metric $g = \omega^k * g_{kl} \otimes_* \omega^l$

Scalar curvature:

$$R = \left\langle \mathcal{G} , Ric \right\rangle_*$$

Einstein's equations

$$\boxed{Ric - \frac{1}{2}\, g * R = 0}$$