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The Standard Model: the electroweak sector

Emilian Dudas[a]

[a] *Centre de Physique Théorique*[1], *Ecole Polytechnique and CNRS, F-91128 Palaiseau, France*

Abstract

This brief introduction to Quantum Field Theory and the Standard Model contains the basic building blocks of perturbation theory in quantum field theory, an elementary introduction to gauge theories and the basic classical and quantum features of the electroweak sector of the Standard Model. Some specific aspects like the effective potential, anomalies and extrapolation of Higgs couplings to high energies are discussed, as well as some mysterious aspects of the Standard Model which motivate new physics constructions.

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Contents

1 Introduction

The development of Quantum Field Theory and the raise of the Standard Model remains as one of the most fascinating adventures of fundamental science of the twenties century. Indeed, despite the seemingly great difference between the strength, action range and the different role played in the birth and the evolution of our universe by the electromagnetic, weak and strong interactions, we know that all three interactions are based on the gauge principle, which seems to be a fundamental principle of nature. Amazingly enough, gauge theories with or without spontaneous symmetry breaking are also renormalizable, in the leading expansion in the dimension of operators in quantum field theory. There is nothing inconsistent from the modern perspective in non-renormalizable theories, the prominent and most important example of this type being Einstein gravity. However, renormalizability renders a theory highly predictive up to high energy scales. This allowed highly precise tests of quantum electrodynamics (QED) like for example the computation of the electron anomalous magnetic moment or the running with the energy of the fine-structure constant. That's why we can talk today about the precision tests of the Standard Model, possible deviations from it, if found experimentally, having to be interpreted unambiguously as signatures of new physics.

These lectures contain an introduction to the basic features of quantum field theory and the electroweak sector of the Standard Model. They are organized as follows. Section 2 introduces symmetries and the Noether theorem. Section 3 introduces perturbation theory, first time-dependent perturbation theory in quantum mechanics, followed by perturbation theory in quantum field theory. Section 4 is an introduction to abelian and non-abelian gauge theories and elements of their quantization. Section 5 describes spontaneous symmetry breaking, Goldstone theorem and the Higgs mechanism. Section 6 introduces the classical aspects of the electroweak sector of the standard sector. Section 7 discusses renormalizability and examples of energy evolution of couplings in the ϕ^4 scalar theory and QED. Section 8 contains some simple applications and constraints coming from global and gauge anomalies. Section 9 enters into the Higgs physics and some theoretical arguments in favor of a light Higgs boson. As well known, Higgs searches are presently the main goal of the Large Hadron Collider (LHC) at CERN. (Very) preliminary LHC results seem to validate the theoretical picture pioneered long-time ago by Higgs and by Brout-Englert [1] and the more recent theoretical arguments pointing in favor of a light scalar Higgs boson. We end up with brief standard arguments in favor of the Standard Model as an effective theory, to be completed beyond some unknown energy scale with an underlying microscopic theory.

2 Fields, Symmetries and the Noether theorem.

Symmetries are fundamental in our understanding in nature. Classic examples are:
- Continuous *spacetime symmetries*, for example space rotations.
- *Discrete symmetries* are fundamental in classification and properties of crystals.
- *Continuous and discrete* internal symmetries in particle physics.

Ex. the *eightfold way* : Flavor $SU(3)_f$, used by M. Gell-Mann in his famous classification of hadrons which also led to the introduction of color as a new quantum number and to the modern theory of strong interactions, the QCD.

The importance of symmetries in nature is to a large extent due to the **Noether theorem** :

To any continuous symmetry of a physical system, it corresponds a conserved current and an associate conserved charge.

Examples of conserved charges associated to continuous symmetries are :

Symmetry	Conserved charge
Time translation	Energy
Space translation	Momentum
Rotations	Angular momentum
Phase rotations wave function	Electric charge

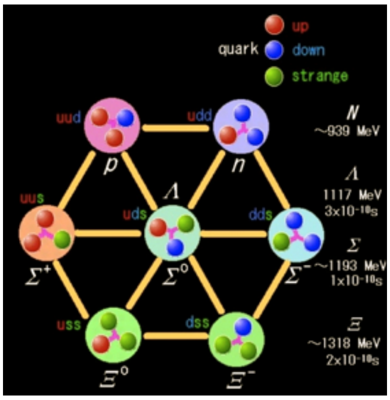


Fig. 1: The flavor $SU(3)_f$ baryon octet. From [2].

For the case of internal symmetries which are our primary goal here, the proof of the Noether theorem goes as follows. Consider a field theory with ϕ denoting collectively all the fields of the theory, of lagrangian $\mathcal{L}(\phi, \partial_m \phi)$. The field transformations generated by infinitesimal parameters α_a

$$\delta\phi \;=\; \alpha_a(x) T^a \phi \;, \tag{1}$$

lead to a new lagrangian

$$\mathcal{L}(\phi, \partial_m \phi) \to \hat{\mathcal{L}}(\phi, \alpha_a, \partial_m \phi, \partial_m \alpha_a) \;. \tag{2}$$

The variation of the action functional $S(\phi, \partial_m \phi)$ under field variations (1) is

$$\begin{aligned} \delta S \;&=\; \int d^4x \left[\frac{\partial \hat{\mathcal{L}}}{\partial \alpha_a} \alpha_a + \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_m \alpha_a)} \partial_m \alpha_a \right] \\ &=\; \int d^4x \left[\frac{\partial \hat{\mathcal{L}}}{\partial \alpha_a} - \partial_m \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_m \alpha_a)} \right] \alpha_a \;, \end{aligned} \tag{3}$$

where to get the result in the last line we performed an integration by parts. By defining the currents

$$J_a^m \;=\; \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_m \alpha_a)} \;, \tag{4}$$

we find that the variation of the action (3) vanishes if

$$\partial_m J_a^m \;=\; \frac{\partial \hat{\mathcal{L}}}{\partial \alpha_a} \;. \tag{5}$$

In the particular case where the field variation is a symmetry of the lagrangian, we immediately find the conservation law

$$\begin{aligned} &\partial_m J_a^m \;=\; 0 \;\Rightarrow\; \frac{dQ_a}{dt} = \int d^3\mathbf{x}\, \partial_m J_a^m \;=\; 0 \;, \\ &\text{where} \qquad Q_a \;=\; \int d^3\mathbf{x}\; J_a^0 \end{aligned} \tag{6}$$

is a conserved charge. In the quantum theory, the charge Q_a is promoted to an operator that generates transformations on the fields

$$\delta\phi \;=\; i\alpha_a \,[Q_a, \phi]. \tag{7}$$

It is straightforward and is left to the reader as an exercise to show that the conserved current can also be computed according to the following formulae

$$\begin{aligned} \delta\hat{\mathcal{L}} &= J_a^m \, \partial_m \alpha_a \ , \quad \text{or} \\ J_a^m &= \frac{\partial \mathcal{L}}{\partial(\partial_m \phi)} \, \frac{\delta \phi}{\delta \alpha_a} \ . \end{aligned} \tag{8}$$

Through the Noether theorem, continuous symmetries lead to conserved charges that are manifest in the spectrum and interactions. As known already from quantum mechanics[2] their study greatly simplifies the dynamics.

As we will see later on, *the local (space-time dependent) symmetries* determine the structure of all the fundamental interactions in nature ! Indeed, all four fundamental interactions, the electromagnetism, the weak and strong forces and (in a somewhat different way) the gravitational one can be found as consequences of local symmetries called **gauge symmetries**. The simplest and most important example is maybe the conservation of the *electric charge*. Applied to the case of a fermion (containing say the electron) Ψ, of lagrangian written explicitly later on in (78) , the symmetry under consideration is simply the phase transformation

$$\Psi \to e^{i\alpha} \Psi \quad , \quad \delta\Psi = i\alpha\Psi \ , \tag{9}$$

which leads to the Noether current and the conserved charge

$$J^m = \bar{\Psi}\gamma^m \Psi \quad , \quad Q = \int d^3x \ \bar{\Psi}\gamma^0\Psi = \int d^3x \ \Psi^\dagger \Psi \ , \tag{10}$$

where J^m is the electromagnetic current of the fermion and Q the electric charge, which becomes an operator in the quantum theory.

In the case of spacetime symmetries, the lagrangian is not really invariant, but it transforms into a total derivative

$$\delta\mathcal{L} \ = \ \partial_m (K_a^m \alpha_a) \ . \tag{11}$$

In this case, the current derived before changes into

$$J_a^m \ = \ \frac{\partial \mathcal{L}}{\partial(\partial_m \phi)} \, \frac{\delta\phi}{\delta\alpha_a} - K_a^m \ . \tag{12}$$

Let us consider as an example spacetime translation acting on a Lorentz scalar

$$x'^m = x^m + a^m \quad , \quad \phi'(x') = \phi(x) \ . \tag{13}$$

At the linear order in a^m, both the scalar and the lagrangian, being Lorentz scalars, transform in the same way

$$\delta\phi \equiv \phi'(x) - \phi(x) \ = \ -a^m \partial_m \phi \ , \ \delta\mathcal{L} \ = \ -a^m \partial_m \mathcal{L} \ . \tag{14}$$

In this case therefore, the corresponding conserved charge is

$$J^m = -\frac{\partial\mathcal{L}}{\partial(\partial_m\phi)} a^n \partial_n \phi + a^m \mathcal{L} \equiv -a_n T^{mn} \ , \tag{15}$$

where

$$T^{mn} = \frac{\partial \mathcal{L}}{\partial(\partial_m \phi)} \partial^n \phi - \eta^{mn} \mathcal{L} \tag{16}$$

is the *energy-momentum tensor*. Noether theorem ensures its the conservation and the existence of the conserved charge

$$\partial_m T^{mn} = 0 \quad , \quad P^m = \int d^3x \, T^{0m} \ , \tag{17}$$

where the conserved charge $P^m = (E, \mathbf{P})$ contains the energy and the total momentum, which are the generators of spacetime translations.

[2]For example the conservation of angular momentum greatly simplifies the study of hydrogen atom.

3 Quantization and perturbation theory.

The second quantization of fields and perturbation theory lead to precise formulae for scattering amplitudes which led to the Feynman diagrams, that are crucial for computing cross sections and other physical observables. The appropriate formalism uses the Heisenberg or interaction picture in quantum mechanics, that we first review, before introducing the corresponding quantum field theory formalism.

3.1 Time-dependent perturbation theory in quantum mechanics.

Let's start from Schrodinger versus interaction/Heisenberg picture in Quantum Mechanics.

$$H \;=\; H_0 \;+\; H_{int} \tag{18}$$

where H_0 is the free hamiltonian and H_{int} is the interaction.
The Schrodinger equation is

$$i\frac{d\,|\Psi_S(t)\rangle}{dt} \;=\; (H_0 \;+\; H_{int})\,|\Psi_S(t)\rangle \tag{19}$$

time dep. ↗ ↖ time-indep. operators.

In the *interaction (or Heisenberg) picture*

$$|\Psi_I(t)\rangle \;=\; e^{iH_0 t}\,|\Psi_S(t)\rangle \quad , \quad H_{int}(t) \;=\; e^{iH_0 t}\,H_{int}(t)\,e^{-iH_0 t} \tag{20}$$

the Schrodinger eq. becomes (**Exercise:**)

$$i\frac{d\,|\Psi_I(t)\rangle}{dt} \;=\; H_{int}(t)\,|\Psi_I(t)\rangle \tag{21}$$

We define the evolution operator $U(t, t_i)$ by

$$|\Psi_I(t)\rangle \;=\; U(t,t_i)|\Psi_I(t_i)\rangle \quad , \quad U(t_i,t_i) \;=\; 1 \tag{22}$$

Ex: Check that U satisfies the eq.

$$i\,\frac{\partial U(t,t_i)}{\partial t} \;=\; H_{int}(t)\,U(t,t_i) \tag{23}$$

It can be shown that (**Ex:**)

$$U(t,t_i) \;=\; T\,e^{-i\int_{t_i}^{t} dt' H_{int}(t')} \tag{24}$$

where the *time-ordered product* of the operators A and B is defined as

$$TA(t_1)B(t_2) \;=\; \theta(t_1-t_2)\;A(t_1)B(t_2) \;+\; \theta(t_2-t_1)\;B(t_2)A(t_1) \tag{25}$$

The S-matrix is defined as

$$S \;=\; \lim_{t\to\infty, t_i\to -\infty} U(t,t_i) \;=\; T\,e^{-i\int dt H_{int}(t)} \tag{26}$$

The states in the far past, before the interaction process are free wave packets and are denoted by $|\; p_1\cdots p_n,\;\text{in}\;\rangle$, where p_i are the momenta of the incident particles. Similarly, the states in the far future, after the interaction process are again free and are denoted by $|\; p'_1\cdots p'_m,\;\text{out}\;\rangle$, where p'_i are the momenta of the scattered particles. The transition amplitudes passing from the initial to the final state is

$$\begin{aligned}
&S_{if} \;=\; \langle\Psi_f|S|\Psi_i\rangle \;=\; \langle p'_1\cdots p'_m,\text{in}|\;S\;|p_1\cdots p_n,\text{in}\rangle \\
&= \langle p'_1\cdots p'_m,\;\text{out}\;|\;p_1\cdots p_n,\;\text{in}\;\rangle \;=\; \text{no interaction term} \\
&+ i\,(2\pi)^4\,\delta^4(\sum_{j=1}^{m} p'_j - \sum_{i=1}^{n} p_i)\;\mathcal{A}_{if}
\end{aligned} \tag{27}$$

The Feynman rules are usually given for the matrix $\mathcal{A}_{if}$.

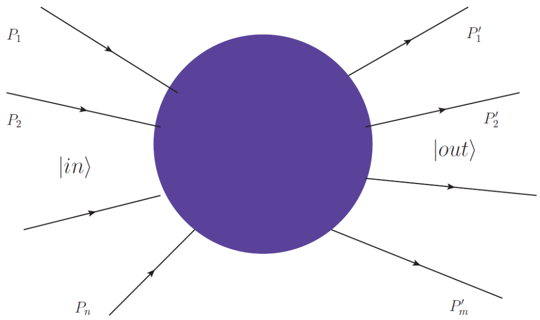


Fig. 2: Scattering amplitude of n initial particles and m final particles. The amplitude of the process is denoted $S_{if} = \langle p'_1 \cdots p'_m,\ \text{out} \,|\, p_1 \cdots p_n,\ \text{in} \,\rangle$

3.2 Quantization of the scalar theory.

Canonical quantization in Quantum field theory uses the Heisenberg (interactive) picture. Let us consider for illustration a scalar theory

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \;=\; \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \\ &= \mathcal{L}_0 + \mathcal{L}_{int} \quad , \quad \text{where} \quad \mathcal{L}_{int} \;=\; -\frac{\lambda}{4!}\phi^4\end{aligned} \tag{28}$$

The metric convention throughout these lectures will be $\eta_{mn} = diag(1,-1,-1,-1)$. The conjugate momentum is $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$ and the hamiltonian

$$\begin{aligned}H &= \int d^3\mathbf{x}\left[\dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathcal{L}\right] \;=\; \int d^3\mathbf{x}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right] \\ &= H_0 + H_{int} \quad , \quad \text{where}\end{aligned} \tag{29}$$

$$\begin{cases} H_0 \;=\; \int d^3\mathbf{x}\ \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2\right] \\ H_{int} \;=\; \int d^3\mathbf{x}\ \frac{\lambda}{4!}\phi^4 \end{cases} \tag{30}$$

Field eqs. called the Klein-Gordon equation and the solutions for the free-field theory are :

$$\begin{aligned}(\Box + m^2)\,\phi(x) \;&=\; 0 \quad \Rightarrow \\ \phi(x) \;&=\; \int \frac{\mathbf{d^3k}}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left(e^{ikx}\,a^\dagger_{\mathbf{k}} \;+\; e^{-ikx}\,a_{\mathbf{k}}\right)\end{aligned} \tag{31}$$

where $k_0 = \omega_k = \sqrt{\mathbf{k}^2 + m^2}$. The solution $\phi(x)$ is the operator in the Heisenberg picture. Quantization proceeds as usual:

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] \;=\; \delta^3(\mathbf{k}-\mathbf{k}') \quad \to \quad [\phi(t,\mathbf{x}, \pi(t,\mathbf{y})] \;=\; i\delta^3(\mathbf{x}-\mathbf{y}) \tag{32}$$

The one-particle states are defined by

$$|\mathbf{k}\rangle \;=\; a^\dagger_{\mathbf{k}}|0\rangle \;\Rightarrow\; \langle\mathbf{k}'|\mathbf{k}\rangle \;=\; \delta^3(\mathbf{k}-\mathbf{k}') \tag{33}$$

and the energy/hamiltonian is

$$H_0 \;=\; \int d^3\mathbf{k}\,\omega_k\,(a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \frac{1}{2}) \tag{34}$$

is one of a collection of quantum oscillators. Therefore (there is by definition no interaction in the asymptotic past and future)

$$\begin{cases} |\psi_i\rangle &= |p_1 p_2 \cdots p_n\rangle = a^\dagger_{\mathbf{p_1}} \cdots a^\dagger_{\mathbf{p_n}} |0\rangle \\ |\psi_f\rangle &= |p'_1 p'_2 \cdots p'_m\rangle = a^\dagger_{\mathbf{p'_1}} \cdots a^\dagger_{\mathbf{p'_m}} |0\rangle \end{cases} \tag{35}$$

3.3 Evolution operator and S-matrix in quantum field theory.

Starting from the scalar field, one can define the free fields before and after the interaction

$$\lim_{t\to-\infty} \phi(x) = Z^{1/2}\phi_{in}(x) \ , \ \lim_{t\to+\infty} \phi(x) = Z^{1/2}\phi_{out}(x) \ . \tag{36}$$

The factors Z in (36) are wave-function normalizations that take into account the possible mismatch in normalization between the (out)ingoing field and the free-field satisfying canonical commutation relations. We define the evolution operator by

$$\phi(x) \ = \ U^{-1}(t) \, \phi_{in}(x) U(t) \ , \tag{37}$$

where $U(t) = U(t, -\infty)$, ϕ_{in} is the incoming (free) field and ϕ is the interacting field. Since ϕ_{in} and ϕ_{out} are free-fields, they generate the same Fock space of states. There should therefore be an unitary operator, the S-matrix, which relates the operators according to

$$\phi_{out}(x) = S^{-1}\phi_{in}(x) S \ , \tag{38}$$

and the corresponding states in the Fock space according to

$$\begin{aligned} |out\rangle &= S^{-1} |in\rangle = S^\dagger |in\rangle \ , \\ |in\rangle &= S \, |out\rangle \ . \end{aligned} \tag{39}$$

Clearly the S-matrix is the infinite-time limit of the evolution operator

$$S = \lim_{t\to\infty} U(t) \ . \tag{40}$$

As in quantum mechanics, we separate the interaction from the free hamiltonian

$$H \ = \ H_0 \ + \ H_{int}(t) \ . \tag{41}$$

The evolution eqs. for the quantum fields are

$$\frac{\partial \phi(x)}{\partial t} \ = \ i \left[H(\phi), \phi(x) \right] \quad , \quad \frac{\partial \phi_{in}\phi(x)}{\partial t} \ = \ i \left[H_0(\phi_{in}), \phi_{in}(x) \right] . \tag{42}$$

By combining (37) and (42), we obtain the eq. satisfied by the evolution operator

$$i\frac{dU}{dt} \ = \ \left(H(\phi_{in}) - H_0(\phi_{in}) \right) U \ = \ H_I(t) \, U \ , \tag{43}$$

where $H_I(t) = H_{int}(\phi_{in}, \pi_{in})$. It is easy to check that the evolution operator satisfies the integral eq.

$$U(t) \ = \ I \ - \ i \int_{-\infty}^{t} dt_1 \, H_I(t_1) \, U(t_1) \ . \tag{44}$$

This eq. can be solved by iteration. It can be shown term by term in the expansion in the interaction that the solution of (44) can be written in the compact elegant form

$$U(t) \ = \ T \, e^{-i \int_{-\infty}^{t} dt' H_I(t')} \ . \tag{45}$$

Consequently, the S-matrix is given by

$$S \ = \ \lim_{t\to\infty} U(t) \ = \ T \, e^{-i \int_{-\infty}^{\infty} dt' H_I(t')} \ = \ T \, e^{i \int d^4x \mathcal{L}_I} \ . \tag{46}$$

Whereas at first sight, the last equality is true only in the absence of derivative interactions $\mathcal{H}_I = -\mathcal{L}_I$, it is actually true in general.

3.4 Cross sections and decay rates

Consider a bunch of particles of type A colliding a target of particles of type B and denote the cross-sectional area common to both bunches by A. If N_A (N_B) denotes the total number of particles A (B) in the bunches, then the interaction cross-section is defined as the ratio

$$\sigma = \frac{\text{Nb. events} \times A}{N_A N_B} . \tag{47}$$

For of an unstable particle A, its decay rate is defined as

$$\Gamma_A = \frac{\text{Nb. decays per unit time}}{N_A} . \tag{48}$$

The matrix element of the interaction part of the S-matrix $S = 1 + i\mathcal{A}$ defines the invariant matrix element

$$\langle p_1 \cdots p_n, in |\, i\mathcal{A}\, |\, q_1 \cdots q_l, in\, \rangle \; = (2\pi)^4 \delta^4 (\sum_k p_k - \sum_r q_r)\; i\mathcal{M}(q_1 \cdots q_l \to \; p_1 \cdots p_n) . \tag{49}$$

The asymptotic states should be constructed in terms of normalized wave packets. For scalars for example they are positive energy solutions $\tilde{\varphi}(x)$ of the Klein-Gordon equation. A two-particle incoming state can then be written

$$|i, in\rangle = \int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} \varphi_1(p_1)\varphi_2(p_2) |p_1, p_2\rangle , \tag{50}$$

where $\varphi(p)$ is the momentum-space Fourier transform

$$\tilde{\varphi}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E} \varphi(p) . \tag{51}$$

The flux of particles in the beam is given by

$$i \int d^3 x \tilde{\varphi}^*(x) \overleftrightarrow{\partial_t} \tilde{\varphi} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E} |\varphi(p)|^2 . \tag{52}$$

The differential cross section two a scattering of two particles into an arbitrary number of final particles $A + B \to p_1 \cdots p_n$ is

$$d\sigma = \frac{1}{4E_A E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(q_A q_B \to \; p_1 \cdots p_n)|^2 (2\pi)^4 \delta^4 (q_A + q_B - \sum_f p_f) , \tag{53}$$

where v_A, v_B are the speed of the particles A, B and $|v_A - v_B|$ is the relative velocity of the two colliding particles in the laboratory frame. The decay rate of an unstable particle A is given by a similar formula

$$d\Gamma = \frac{1}{2m_A} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(m_A \to \; p_1 \cdots p_n)|^2 (2\pi)^4 \delta^4 (q_A - \sum_f p_f) . \tag{54}$$

The amplitude of a proces with a virtual exchange of an unstable particle of momentum p is given by a relativistic extension of the Breit-Wigner formula

$$\mathcal{M} \; \sim \; \frac{1}{p^2 - m^2 + im\Gamma} , \tag{55}$$

where Γ is the total decay width of the particle. The Breit-Wigner formula correctly transform the infinite peak on the mass shell $p^2 = m^2$ into a gaussian bump and is crucial in order to correctly interpret a resonance in experimental data.

3.5 Reduction formula, perturbation theory and Feynman diagrams.

Feynman rules and perturbation theory follow from the expansion in powers of the interaction of S-matrix elements

$$\langle p_1 \cdots p_n, in|\, S \,|\, q_1 \cdots q_l, in\,\rangle \;=\; \langle 0|a_{\mathbf{p'_m}} \cdots a_{\mathbf{p'_1}}\; T\, e^{i\int d^4x \mathcal{L}_{int}(x)} a^\dagger_{\mathbf{p_1}} \cdots a^\dagger_{\mathbf{p_n}}\, |0\rangle\,. \tag{56}$$

A very important formula in S-matrix perturbation theory is the reduction or the LSZ (Lehmann-Symanzik-Zimmermann) formula, which relates S-matrix elements to the time-ordered Green functions

$$\begin{aligned} &\langle p_1 \cdots p_n, out|\, q_1 \cdots q_l, in\rangle \;=\; \langle p_1 \cdots p_n, in|\, S \,|\, q_1 \cdots q_l, in\,\rangle \\ &=\; \text{disconnected terms} \;+\; (iZ^{-1/2})^{n+l} \times \\ &\times \int d^4y_1 \cdots d^4x_l\, e^{i(\sum_k p_k y_k - \sum_r q_r x_r)}\, (\Box_{y_1} + m^2) \cdots (\Box_{x_1} + m^2) \langle 0|\, T\phi(y_1) \cdots \phi(x_l)\, |0\rangle \end{aligned} \tag{57}$$

where Z is the wave-function renormalization for the scalar field. The asymptotic Fock space defines unambigously the vacuum state

$$|0, in\rangle \;=\; |0, out\rangle \;=\; |0\rangle \tag{58}$$

The central figures in perturbation theory are therefore the *Green functions*

$$G^{(n)}(x_1 \cdots x_n) \;=\; \langle 0|\, T\phi(y_1) \cdots \phi(x_n)\, |0\rangle \tag{59}$$

The Green functions of the interactive field ϕ can be expressed in terms of Green functions of the free-field ϕ_{in} via the crucial formula (see for ex. [3, 52])

$$G^{(n)}(x_1 \cdots x_n) \;=\; \frac{\langle 0|\, T\phi_{in}(x_1) \cdots \phi_{in}(x_n)\, e^{i\int d^4x \mathcal{L}_{int}[\phi_{in}]}\, |0\rangle}{\langle 0|\, T\, e^{i\int d^4x \mathcal{L}_{int}[\phi_{in}]}\, |0\rangle}\,. \tag{60}$$

Green functions can be elegantly captured by a generating functional, defined by coupling the scalar to an external source $J(x)$, by changing the lagrangian $\mathcal{L} \to \mathcal{L} + J\phi$. In this case

$$Z(J) \;=\; e^{iW(J)} \;=\; \langle 0|T e^{i\int d^4x\; \phi(x)J(x)}|0\rangle\,. \tag{61}$$

It can be shown that

$$G^{(n)}(x_1 \cdots x_n) = \frac{1}{i^n} \frac{\delta^n W(J)}{\delta J(x_1) \cdots \delta J(x_n)}|_{J=0} \tag{62}$$

are connected Green functions, generated therefore by the generating functional $W(J)$. Conversely, W can be expanded in a power series

$$W(J) = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n\, G^{(n)}(x_1 \cdots x_n) J(x_1) \cdots J(x_n)\,. \tag{63}$$

Another important notion is **normal ordering**. A normal ordered operator $:\mathcal{O}:$ is defined such that all creation operators are on the left and all annihilation operators are on the right. By construction then its vev vanishes $\langle :\mathcal{O}: \rangle = 0$. G. Wick found an elegant way to express free-field Green functions in terms of normal-ordered products, by the so-called **Wick theorem**. The simplest example is the two-point function

$$T\phi_{in}(x)\phi_{in}(y) \;=: \phi_{in}(x)\phi_{in}(y) :\; + D_F(x-y)\,, \tag{64}$$

where $D_F(x-y)$ is the Feynman propagator. An explicit computation gives

$$D_F(x-y) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} \{\theta(x_0 - y_0)\, e^{-ik(x-y)} \;+\; \theta(y_0 - x_0)\, e^{ik(x-y)}\}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)} . \tag{65}$$

The $i\epsilon$ prescription in the Feynman propagator has the property of propagating the positive frequencies into the future and the negative frequencies into the past. This is precisely what will be needed later on in order to capture both particles and antiparticles propagation in a causal way. Wick theorem can be generalized to a time-ordered product of an arbitrary number of fields

$$T\phi_{in}(x_1)\phi_{in}(x_2)\cdots\phi_{in}(x_n) \;=: \phi_{in}(x_1)\phi_{in}(x_2)\cdots\phi_{in}(x_n) : \; + \text{all possible contractions} . \tag{66}$$

Let us now discuss the *Feynman diagrams* for the simplest ϕ^4 theory, with

$$\mathcal{L}_{int} \;=\; -\frac{\lambda}{4!}\,\phi^4 . \tag{67}$$

The Feynman rules are usually formulated in the momentum space :

$$G(p_1\cdots p_n) \;=\; \int d^4x_1\cdots d^4x_n \, e^{i\sum_i p_i x_i}\, G(x_1\cdots x_n) . \tag{68}$$

Applying perturbation theory (57), we obtain the following **Feynman rules** :

• associate to each propagator the factor $\frac{i}{p^2-m^2+i\epsilon}$.

• to each vertex the factor $-i\lambda$.

• impose momentum conservation at each vertex.

• integrate over undetermined internal momenta k, $\int \frac{d^4k}{(2\pi)^4}$.

• each diagram is to be divided by a symmetry factor, equal to the number of ways of interchanging components without changing the diagram.

• sum the contributions of all topologically distinct *connected* diagrams.

It can also be shown from (57) that the denominator cancels precisely all non-connected diagrams in the Feynman diagrams of the Green functions.

Perturbation theory is now one of the cornerstones of QFT. The anomalous magnetic moment of the electron was computed for the first time by Schwinger at one-loop in 1948 [13] (the factor below, $\frac{\alpha}{2\pi}$, is engraved on Schwinger's tombstone). Today it is known up to four-loops !

$$\begin{aligned} a_e &= \frac{g_e-2}{2} \;=\; \frac{\alpha}{2\pi} \;+\cdots \\ a_e^{\rm exp} &\;=\; (1159652185.9 \pm 3.8)\times 10^{-12} , \\ a_e^{\rm th} &\;=\; (1159652175.9 \pm 8.5)\times 10^{-12} , \end{aligned} \tag{69}$$

where g_e is the gyromagnetic factor of electron coupling to a magnetic field. The theoretical prediction agrees with the experimental measurements in (69) up to the eight digit ! There are however *still mysteries* in perturbation theory. For example, for the muon magnetic moment, the measured value at BNL disagrees by $3.4\,\sigma$ from the theoretical SM calculation

$$\begin{aligned} a_\mu^{\rm th} &\;=\; a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm had} \\ a_\mu^{\rm exp} &\;\simeq\; 0,00116592089 \end{aligned}$$

In this case, it is likely that the hadronic contribution is not known accurately enough, since the muon mass is much closer to the hadronic contributions compared to the electron one. This is a very hot research topic nowdays, since any real disagreement could be a hint for new physics contributions coming from virtual loops of new particles.

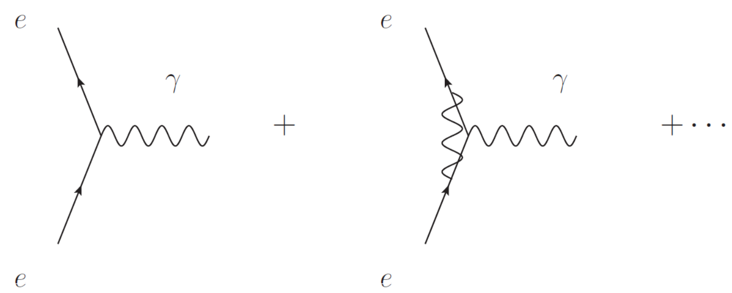


Fig. 3: Simplest Feynman diagrams contributions to the electron magnetic moment. The agreement between perturbative QED computations and the experimentally measured value agree up to the eight digit.

3.6 Ward identities and low energy theorems

Let us consider a current

$$J_m = \frac{\delta \mathcal{L}}{\delta(\partial^m \phi)} \delta\phi \, , \tag{70}$$

where the field transformation is generated by the Noether charge associated to the current J

$$\delta\phi(x) = i[Q, \phi(x)] \, . \tag{71}$$

Let's furthermore consider the possibility that the current is conserved or not

$$\partial^m J_m(x) \; \equiv \; \Delta(z) \, . \tag{72}$$

As shown in the previous sections, the relevant objects in perturbation theory are time-ordered correlation functions of the quantum fields. In this case, it is straightforward to prove the following equality, called the Ward identity

$$\begin{aligned} &\frac{\partial}{\partial z_m} \langle 0|T J_m(z)\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle = \langle 0|T\Delta(z)\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \\ &-i\delta^4(z-x_1)\langle 0|T\delta\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle - i\delta^4(z-x_2)\langle 0|T\phi(x_1)\delta\phi(x_2)\cdots\phi(x_n)|0\rangle - \cdot \end{aligned} \tag{73}$$

If one takes the Fourier transform of (73), one gets

$$\begin{aligned} &ik_m \int d^4z \, e^{-ikz} \langle 0|T J_m(z)\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle = \int d^4z \, e^{-ikz} \langle 0|T\Delta(z)\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \\ &-ie^{-ikx_1}\langle 0|T\delta\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle - ie^{-ikx_2}\langle 0|T\phi(x_1)\delta\phi(x_2)\cdots\phi(x_n)|0\rangle - \cdots \, . \end{aligned} \tag{74}$$

In the zero-momentum limit limit one gets

$$\begin{aligned} &\int d^4z \; \langle 0|T\Delta(z)\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \; = \\ &i \; \langle 0|T\delta\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle + i \; \langle 0|T\phi(x_1)\delta\phi(x_2)\cdots\phi(x_n)|0\rangle + \cdots \\ &= i \, \delta \; \langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \, . \end{aligned} \tag{75}$$

Such relations are called low-energy theorems.

3.7 Fermions and the quantization of the Dirac field

Relativistic fermions satisfying the Pauli principle are described by *spinors* in quantum field theory. In particular, the relativistic spin $1/2$ fermion is described by a four component spinor Ψ via the Dirac equation

$$(i\gamma^m\partial_m - M)\Psi \ = 0 \ , \tag{76}$$

where γ_m are the 4×4 Dirac matrices satisfying the Clifford algebra

$$\{\gamma^m, \gamma^n\} \ = \ 2\eta_{mn} \ . \tag{77}$$

The Lagrangian giving the Dirac equation is

$$\mathcal{L}_0 \ = \ \bar{\Psi}(i\gamma^m\partial_m - M)\Psi \ , \tag{78}$$

where $\bar{\Psi} = \Psi^\dagger\gamma^0$. A particular role is played by fermions which are eigenstates of the chirality operator, satisfying

$$\begin{aligned} &\gamma^5 \ = \ i\gamma_0\gamma_1\gamma_2\gamma_3 \quad , \quad (\gamma^5)^2 \ = \ 1 \ , \\ &\{\gamma^m, \gamma^5\} \ = \ 0. \end{aligned} \tag{79}$$

It is then possible to define left and right-handed chirality fermions

$$\begin{aligned} \gamma^5\Psi_L \ &= \ -\Psi_L \quad , \quad \Psi_L \ = \ \frac{1-\gamma^5}{2} \ \Psi \ , \\ \gamma^5\Psi_R \ &= \ \Psi_R \quad , \quad \Psi_R \ = \ \frac{1+\gamma^5}{2} \ \Psi \ . \end{aligned} \tag{80}$$

In terms of the left/right chirality fermions, the Dirac lagrangian is written

$$\mathcal{L}_0 \ = \ \bar{\Psi}_L i\gamma^m\partial_m\Psi_L + \bar{\Psi}_R i\gamma^m\partial_m\Psi_R - M(\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L) \ , \tag{81}$$

whereas the Dirac equation can be split into two equations

$$i\gamma^m\partial_m\Psi_L - M\Psi_R \ = \ 0 \ , \ i\gamma^m\partial_m\Psi_R - M\Psi_L \ = \ 0 \ . \tag{82}$$

A particularly convenient basis for gamma matrices, when dealing with chiral fermions, is the chiral representation, for which

$$\begin{aligned} &\gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix} \quad , \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \\ &\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad C = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} \ , \end{aligned} \tag{83}$$

where C is the charge conjugation matrix satisfying

$$C^{-1}\,\gamma_m\,C = -\,\gamma_m^T \quad , \quad C \ = \ -C^{-1} \ = \ -C^T \ = \ -C^\dagger \tag{84}$$

The Dirac lagrangian and Dirac equation have two different type of symmetries, a vector symmetry of parameter α and an axial-symmetry of parameter β

$$\begin{aligned} &\text{vector}\ : \qquad \Psi_L \to e^{i\alpha}\Psi_L \quad , \quad \Psi_R \to e^{i\alpha}\Psi_R \ , \\ &\text{axial}\ : \qquad \Psi_L \to e^{i\beta}\Psi_L \quad , \quad \Psi_R \to e^{-i\beta}\Psi_R \ . \end{aligned} \tag{85}$$

The axial transformation can also be written in the form

$$\Psi_R \to e^{i\beta\gamma_5}\Psi_R \tag{86}$$

Whereas the vector symmetry is exact for any value of the mass M and through Noether theorem is responsible for the charge conservation, the axial symmetry is broken by the mass term and is therefore exact clasically only in the massless limit. As we will see in Section 6, in Nature left and right chirality fermions have different interactions. This is related to the parity violation in the weak interactions and it is at the heart of the construction of the Standard Model.
There are two different type of fermions that could exist in nature. The fermions charged under gauge symmetries are of Dirac type, i.e. their mass (eventually after symmetry breaking, as it will be the case in the Standard Model) is of Dirac type (81). For fermions uncharged under gauge symmetries, they can be of *Majorana* type. In this case, the charge conjugate fermion

$$\Psi^c \; = \; C \bar{\Psi}^T \; , \tag{87}$$

is self-conjugate $\Psi^c = \Psi$, i.e. the fermion is its own antiparticle. In this case, the mass of the fermion can be written as

$$\mathcal{L}_M \; = \; - \frac{M}{2} \Psi^T C \Psi + \text{h.c.} \; . \tag{88}$$

It is not yet known if there exist Majorana fermions in nature. One natural possibility are the neutrinos.

According to perturbation theory, we start from the free Dirac lagrangian

$$\mathcal{L}_0 \; = \; \bar{\Psi} (i \gamma^m \partial_m - M) \Psi \; . \tag{89}$$

Conjugate momentum is $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = i \Psi^\dagger$. The free-field hamiltonian is then

$$H_0 \; = \; \int d^3\mathbf{x} \; \bar{\Psi} (i \gamma \nabla + M) \Psi \; = \; \int d^3\mathbf{x} \; \Psi^\dagger (i \alpha \nabla + \beta M) \Psi \; , \tag{90}$$

where $\gamma = \beta\alpha$, $\gamma_0 = \beta$ and in the last paranthesis we can recognize the Dirac hamiltonian of relativistic quantum mechanics. The solutions of the Dirac equation (89) are of the form

$$\Psi(x) \; = \; \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2} \sqrt{2\omega_k}} \sum_{s=1,2} \Big[\, e^{-ikx} \, a^s_{\mathbf{k}} \, u^s(k) \; + \; e^{ikx} \, b^{s,\dagger}_{\mathbf{k}} \, v^s(k) \, \Big] \; , \tag{91}$$

where $u^s(k)$ ($v^s(k)$) are positive (negative) frequency solutions of the Dirac eq.

$$(\gamma^m k_m - M) \, u^s(k) \; = \; 0 \quad , \quad (\gamma^m k_m + M) \, v^s(k) \; = \; 0 \; . \tag{92}$$

The Dirac eq. have two independent solutions $s = 1, 2$. The correct quantization for fermions uses anti-commutators

$$\{ \Psi_\alpha(t, \mathbf{x}), \Psi^\dagger_\beta(t, \mathbf{y}) \} \; = \; \delta_{\alpha\beta} \, \delta^3(\mathbf{x} - \mathbf{y}) \; , \tag{93}$$

all the other anticommutators being zero. This defines the anticomutation relations

$$\{ a^r_{\mathbf{p}}, a^{s,\dagger}_{\mathbf{q}} \} \; = \; \{ b^r_{\mathbf{p}}, b^{s,\dagger}_{\mathbf{q}} \} = \; \delta^{rs} \delta^3(\mathbf{p} - \mathbf{q}) \; . \tag{94}$$

The vacuum is defined by $a^r_{\mathbf{p}} |0 >= b^r_{\mathbf{p}} |0 >= 0$, whereas the hamiltonian is given by

$$H_0 \; = \; \int \; d^3\mathbf{k} \sum_s \omega_k \, \Big[a^{s,\dagger}_{\mathbf{k}} a^s_{\mathbf{k}} \; + \; b^{s,\dagger}_{\mathbf{k}} b^s_{\mathbf{k}} \Big] \; . \tag{95}$$

Notice that if the theory would have been quantized with commutators, the contribution of the b oscillators would have been of opposite sign and the hamiltonian would have been unbounded from below. The electric charge operator can be defined as in (120) and equals

$$Q \; = \; \int \; d^3\mathbf{k} \sum_s \Big[a^{s,\dagger}_{\mathbf{k}} a^s_{\mathbf{k}} \; - \; b^{s,\dagger}_{\mathbf{k}} b^s_{\mathbf{k}} \Big] \; . \tag{96}$$

By defining also the helicity operator, it can shown that:
- $a_{\mathbf{k}}^{s,\dagger}$ creates fermions of energy ω_k, momentum $\mathbf{k}$, electric charge $+1$ (in units of the electron electric charge), helicity left (right) for $s = 1$ ($s = 2$).
- $b_{\mathbf{k}}^{s,\dagger}$ creates antifermions of energy ω_k, momentum $\mathbf{k}$, electric charge -1 and helicity right (left) for $s = 1$ ($s = 2$).
Similarly for the scalars case, there is a reduction/LSZ formula (56) and perturbative expansion (57) for the Green functions. The simplest and most important Green function is the fermionic Feynman propagator

$$\begin{aligned} S_F(x-y) \ &= \ \langle 0 | \, T\Psi(x)\bar{\Psi}(y) \, |0\rangle \ = \ \theta(x^0 - y^0)\langle 0|\Psi(x)\bar{\Psi}(y)|0\rangle \\ -\theta(y^0 - x^0)\langle 0|\bar{\Psi}(y)\Psi(x)|0\rangle &= \int \frac{d^4k}{(2\pi)^4} \frac{i(\gamma^m k_m + M)}{k^2 - m^2 + i\epsilon} \, e^{-ik(x-y)} \ . \end{aligned} \tag{97}$$

3.8 Quantization of the electromagnetic field

The quantization of the free electromagnetic field is subtler than for the case of scalars and fermions. Indeed, starting from the Maxwell lagrangian

$$\mathcal{L} \ = \ -\frac{1}{4} F_{mn} F^{mn} \ , \tag{98}$$

the conjugate momentum is

$$\pi^m = \frac{\partial \mathcal{L}}{\partial \dot{A}_m} \ = \ F^{m0} \quad \Rightarrow \quad \pi^0 = 0 \tag{99}$$

and we cannot impose canonical commutation relations. The problem can be avoided by using the non-covariant gauges, like the Coulomb gauge (div$\mathbf{A} = 0$), but it is preferable to maintain manifest Lorentz covariance. The standard option is to modify the lagrangian by adding a *gauge-fixing term* and changing the lagrangian to

$$\mathcal{L} \ = \ -\frac{1}{4} F_{mn} F^{mn} \ - \ \frac{1}{2\xi} (\partial_m A^m)^2 \ , \tag{100}$$

where ξ is a real arbitrary (and unphysical) parameter. In this case the field eqs. become

$$\Box A_m \ - \ (1 - \frac{1}{\xi}) \partial_m (\partial A) \ = \ 0 \tag{101}$$

and the canonical momentum $\pi^0 \ = \ -\frac{1}{\xi}(\partial_m A^m)$ does not vanishes anymore. The propagator of the photon including the gauge fixing is found by inverting the quadratic part of the lagrangian (101)

$$\mathcal{L} \ = \ \frac{1}{2} A^m \left[\eta_{mn} \Box - (1 - \frac{1}{\xi}) \partial_m \partial_n \right] A^n \ . \tag{102}$$

The result is

$$\Delta_{mn} \ = \ -i \frac{\eta_{mn} - (1 - \xi) \frac{k_m k_n}{k^2}}{k^2 + i\epsilon} \ . \tag{103}$$

The commonly used gauges in Feynman diagram computations are $\xi = 0$ (Landau gauge) and $\xi = 1$ (Feynman gauge).

Observation: Field eqs. imply $\Box \, (\partial A) = 0$, which suggests that we could impose the Lorentz condition $\partial A = 0$. This is however incompatible with canonical quantization, since $\pi^0 \sim \partial A$. The condition can be only be imposed on physical states $|\psi_{ph}\rangle$

$$\langle \psi_{ph} | \partial_m A^m | \psi_{ph} \rangle \ = \ 0 \ . \tag{104}$$

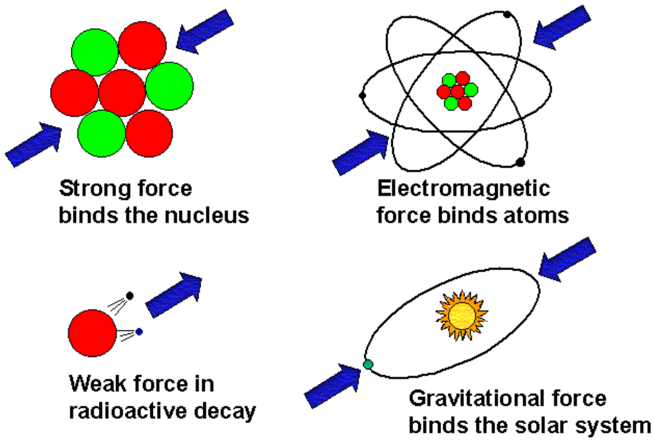


Fig. 4: The four fundamental interactions in nature. From [14].

It can be shown that physical results are independent of ξ. A convenient choice for canonical quantization is $\xi = 1$ (Feynman gauge), in which case the electromagnetic field become a collection of four Klein-Gordon fields $\Box A_m = 0$. In this case, it can be expanded in plane waves according to

$$A_m(x) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}\sqrt{2\omega_k}} \sum_{r=0}^{3} \Big[\, e^{-ikx}\; a^r_{\mathbf{k}}\; \epsilon^r_m(k) \;+\; e^{ikx}\; a^{r,\dagger}_{\mathbf{k}}\; \bar{\epsilon}^r_m(k) \,\Big] \;, \tag{105}$$

where here $\omega_k = |\mathbf{k}| = k^0$ and $\epsilon^r_m(k)$ are the polarization vectors. Canonical quantization in this case goes as follows

$$[A_m(t,\mathbf{x}), \pi_n(t,\mathbf{y})] = -i\eta_{mn}\delta^3(\mathbf{x}-\mathbf{y}) \quad \Rightarrow \quad [A_m(t,\mathbf{x}), \dot{A}_n(t,\mathbf{y})] = -i\eta_{mn}\delta^3(\mathbf{x}-\mathbf{y}) \;. \tag{106}$$

The commutation relations for the creation/annihilation operators then follow

$$[a^r_{\mathbf{k}}, a^{s,\dagger}_{\mathbf{q}}] \;=\; -\eta^{rs}\; \delta^3(\mathbf{k}-\mathbf{q}) \;. \tag{107}$$

Finally, the Feynman propagator in this gauge is

$$\langle 0|T A_m(x) A_n(y)|0\rangle \;=\; -\,\eta_{mn} D_F(x-y)|_{M=0} \;=\; -i\eta_{mn} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2+i\epsilon} \;. \tag{108}$$

4 Gauge theories.

The four fundamental interactions in nature have a common feature: they are *gauge interactions.* We will discuss here the internal symmetries which describe the electromagnetic, weak and strong interactions and elements of their quantization.

4.1 Gauge invariance of Schrödinger eq.

The simplest example of gauge symmetry arises in the description of particle of mass m and charge q in quantum mechanics. The hamiltonian is

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qV \;, \tag{109}$$

where the vector $\mathbf{A}$ and the scalar V potential are related to the electric/magnetic fields via

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A} \;. \tag{110}$$

The Maxwell eqs. are invariant under the gauge transformations

$$\mathbf{A}' = \mathbf{A} + \nabla\alpha \; , \; V' = V - \frac{\partial\alpha}{\partial t} \; . \tag{111}$$

The Schrödinger eq. is *covariant*, with $H = H(\mathbf{A}, V)$, $H' = H(\mathbf{A}', V')$

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi \; \rightarrow \; i\hbar\frac{\partial\Psi'}{\partial t} = H'\Psi' \tag{112}$$

if under the gauge transformations (109), the wave function transforms as

$$\Psi'(\mathbf{r}, t) \; = \; e^{\frac{iq\alpha}{\hbar}} \; \Psi(\mathbf{r}, t) \; . \tag{113}$$

• Notice that the mean value of any physically measurable quantity is *gauge invariant*; for ex. $P(\mathbf{r}) = |\Psi|^2 = |\Psi'|^2$.
Homework: Defining the velocity operator
$\mathbf{v} = \frac{1}{m}(\mathbf{p} - q\mathbf{A})$, check that $\langle\Psi|\mathbf{v}|\Psi\rangle = \langle\Psi'|\mathbf{v}'|\Psi'\rangle$.

Gauge principle : Postulate that physical laws are invariant under (111)+ (113). In this case, it can be proven that the hamiltonian *is uniquely determined* to be (109). Eqs. (113) + (111) define an $U(1)$ transformation.
Therefore, $U(1)$ gauge invariance determines the electromagnetic interaction.

4.2 From Dirac and Maxwell eqs. to QED.

Maxwell eqs. in terms of $A_m = (\mathbf{A}, V)$ are invariant under the *gauge transformations*

$$A_m \rightarrow A'_m = A_m - \partial_m\alpha \; . \tag{114}$$

Gauge invariance postulate : the physics is invariant under (114), supplemented with the phase transformation

$$\Psi(x) \rightarrow \Psi'(x) = e^{iq\alpha(x)}\Psi(x) \; . \tag{115}$$

Then Dirac eq. is not invariant under (115) unless we replace the derivative with the *covariant derivative*

$$D_m\Psi \equiv (\partial_m + iqA_m)\Psi \; \rightarrow \; (D_m\Psi)' = (\partial_m + iqA'_m)\Psi' = e^{iq\alpha(x)}D_m\Psi(x) \; . \tag{116}$$

Dirac equation in an electromagnetic field becomes therefore

$$(i\gamma^m D_m - M)\Psi = (i\gamma^m\partial_m - q\gamma^m A_m - M)\Psi = 0 \; . \tag{117}$$

The Dirac and Maxwell eqs. can be derived from the lagrangian density

$$\mathcal{L}_{QED} \; = \; \bar{\Psi}(i\gamma^m D_m - M)\Psi - \frac{1}{4}F_{mn}^2 \; . \tag{118}$$

The coupled Euler-Lagrange field eqs. are then (117), plus

$$\partial^m F_{mn} = g\bar{\Psi}\gamma_n\Psi \equiv j_n \; , \tag{119}$$

where j_n is the *electromagnetic current* of the charged fermion. Notice that j_n is precisely the current constructed according to the Noether procedure described previously. From (119) we can derive the *charge conservation* law

$$\partial^m j_m = 0 \rightarrow \frac{dQ}{dt} = \int d^3\mathbf{x} \, \partial^m j_m = 0 \; , \; \text{where} \quad Q \; = \; \int d^3\mathbf{x} \; j_0(\mathbf{x}) \; . \tag{120}$$

Let us make some comments :
• The massless photon has *two propagating degrees of freedom*.
• A photon mass $\mathcal{L}_{mass} = \frac{M_A^2}{2} A_m^2$ *breaks gauge invariance* and describes *three degrees of freedom*.
• The propagator of a massive photon is found from inverting the free lagrangian

$$\begin{aligned}\mathcal{L}_{\text{Proca}} &= -\frac{1}{4}F_{mn}^2 + \frac{M_A^2}{2}A_m^2 = \frac{1}{2}A^m[g_{mn}(\Box + M_A^2) - \partial_m\partial_n]A^n , \\ &\Delta_{mn}^{-1}(x-y) = -i[g_{mn}(\Box + M_A^2) - \partial_m\partial_n]\delta^4(x-y) \ . \end{aligned} \tag{121}$$

Therefore, in momentum space (**Homework**)

$$\Delta^{mn}(k) \ = \ -i \ \frac{g^{mn} - \frac{k^m k^n}{M_A^2}}{k^2 - M_A^2} \ . \tag{122}$$

Notice that due to the current conservation $\partial^m j_m = 0$, the longitudinal polarization does not contribute to amplitudes. Therefore, the UV properties of the massless and massive photon theories are *the same*. On the other hand, experimentally the photon is massless to a high accuracy. Indeed, the present experimental limit on the photon mass is $m_\gamma \leq 10^{-18}$ eV.

Finally, we can give **the Feynman rules for QED** :
- associate to each fermion propagator of momentum p the factor $\frac{i(\gamma^m p_m + M)}{p^2 - M^2 + i\epsilon}$.
- to each photon propagator of momentum p (in the $\xi = 1$ gauge) the factor $\frac{-i\eta_{mn}}{p^2 - M^2 + i\epsilon}$.
- to each vertex the factor $iQe\gamma^m$, where $Q = -1$ for the electron.
- to each external initial fermion the factor $u^s(p)$.
- to each external final fermion the factor $\bar{u}^s(p)$.
- to each external initial antifermion the factor $\bar{v}^s(p)$.
- to each external final antifermion the factor $v^s(p)$.
- to each external initial photon the factor $\epsilon_m(p)$.
- to each external final photon the factor $\bar{\epsilon}_m(p)$.

The reader can find more about the historical rise of QED in [9].

4.3 Massive vector fields: Proca and Stueckelberg theories

Independently of H. Yukawa that proposed in 1935 the theory of nuclear forces mediated by scalar mesons exchanges, A. Proca proposed in 1936 a theory of nuclear forces where the mediatior is a massive vector field. The lagrangian he proposed, called Proca lagrangian [10] is

$$\mathcal{L}_{\text{Proca}} \ = \ - \ \frac{1}{4}F_{mn}^2 \ + \ \frac{M_A^2}{2}A_m^2 \tag{123}$$

leads to the field equations

$$\partial^m F_{mn} + M_A^2 A_n \ = \ (\Box + M_A^2)A_n - \partial_n(\partial A) \ = \ 0 \ . \tag{124}$$

Taking a further derivative on the field eqs. (124) we find as a consequence

$$\partial A \ \equiv \ \partial^m A_m \ = \ 0 \ . \tag{125}$$

Notice that, unlike the Maxwell lagrangian in which (125) is not a consequence of field dynamics, but one of the possible gauge choices, in the Proca case its interpretation is different. Gauge symmetry is explicitly broken in the Proca lagrangian by the mass term M_A, whereas (125) is just a consequence of field eqs. Because of this, the number of degrees of freedom of a massive vector field is three, compared to the two degrees of freedom of the photon. Notice that by using (125) one finds that the massive vector field satisfies the same Klein-Gordon equation (31) than the massive spin-zero scalar field

$$(\Box + M_A^2)\, A_m \;=\; 0\;. \tag{126}$$

Quantization of the massive vector field is therefore very similar to the one of the massive scalar described in Section 3.2.

That there is another formulation of the massive vector theory introduced by E. Stueckelberg in 1938 [11], which is manifestly gauge invariant and illuminating from the modern point of view. It uses, in addition to the gauge field, an additional pseudo-scalar field a and is given by

$$\mathcal{L}_{\rm Stueckelberg} \;=\; -\,\frac{1}{4}F_{mn}^2 \;+\; \frac{1}{2}(\partial_m a + M_A A_m)^2\;. \tag{127}$$

Notice that the lagrangian (127) is invariant under the gauge transformations

$$A'_m(x) = A_m(x) - \partial_m \alpha(x)\;,\; a' = a + M_A\,\alpha(x)\;. \tag{128}$$

Notice that due to gauge invariance, we can choose a gauge $\alpha = -(a/M_A)$ where the Stueckelberg scalar is set to zero and the Stueckelberg lagrangian reduces to the Proca one. This gauge is called the unitary gauge. In modern terms, the Stueckelberg scalar is absorbed and provides the longitudinal component of the gauge field. This is precisely what is realized in the Higgs mechanism, that we will describe later on.

4.4 Non-abelian gauge theories

$U(1)$ is a particular case of *unitary abelian transformations*. Another case of particular interest are the non-abelian transformations.

$SU(n)$ transformations are described by $n \times n$ matrices U, satisfying

$$U^\dagger U = U U^\dagger = I \quad , \quad det\, U = 1\;. \tag{129}$$

The simplest case is $SU(2)$, proposed by Yang and Mills in 1954 [15]. Its simplest representation is a *doublet*

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\;,\; \Psi' = U(\theta)\Psi\;,\; \text{where } U(\theta) = e^{\frac{i}{2}g\theta_a\tau_a}\;, \tag{130}$$

where τ_a are the Pauli matrices and g is the $SU(2)$ gauge coupling. It turns out that *the number of gauge bosons W_m^a equals the number of generators* (three for $SU(2)$). The most compact notation introduces a matrix

$$W_m = W_m^a\frac{\tau_a}{2} = \begin{pmatrix} W_m^3 & W_m^1 - iW_m^2\\ W_m^1 + iW_m^2 & -W_m^3\end{pmatrix} \equiv \begin{pmatrix} W_m^3 & \sqrt{2}W_m^+\\ \sqrt{2}W_m^- & -W_m^3\end{pmatrix} \tag{131}$$

Homework : show that

$$\begin{aligned} &D_m\Psi \equiv (\partial_m - igW_m)\Psi \to (D_m\Psi)' = UD_m\Psi\;,\\ &\text{if } W_m \to W'_m = UW_mU^{-1} - \frac{i}{g}(\partial_m U)U^{-1}\;. \end{aligned} \tag{132}$$

The infinitesimal gauge variation in component form is

$$\delta W_m^a = D_m\theta^a \equiv \partial_m\theta^a + g\epsilon_{abc}W_m^b\theta^c\;. \tag{133}$$

The field strength is built from

$$[D_m, D_n] \;=\; -i\,g\,F_{mn}\;. \tag{134}$$

Homework : Show that

$$F_{mn} = \partial_m W_n - \partial_n W_m - ig[W_m, W_n] \ , \ F_{mn} \to F'_{mn} = U F_{mn} U^{-1} \ . \tag{135}$$

For $SU(2)$ this implies (**homework :**)

$$F^a_{mn} = \partial_m W^a_n - \partial_n W^a_m + g\epsilon_{abc} W^b_m W^c_n \ . \tag{136}$$

The Yang-Mills lagrangian is finally given by

$$\begin{aligned}\mathcal{L}_{YM} = -\frac{1}{4} F^a_{mn} F^{a,mn} = -\frac{1}{4}(\partial_m W^a_n - \partial_n W^a_m)^2 \\ -\frac{g}{2}\epsilon_{abc}\partial_m W^a_n W^{b,m} W^{c,n} - \frac{g^2}{4}\epsilon_{abc}\epsilon_{ade} W^b_m W^c_n W^{d,m} W^{e,n} \ .\end{aligned} \tag{137}$$

For a general Yang-Mills theory of generators T_a, the generalisation of the antisymmetric tensor ϵ_{abc} are the structure constants f_{abc} defined according to

$$[T_a, T_b] \ = \ i \, f_{abc} \, T_c \ . \tag{138}$$

The lagrangian and the field strength have still the forms (136)-(137), with the replacement $\epsilon_{abc} \to f_{abc}$.

Notice that *non-abelian gauge bosons have self-interactions, unlike the photon !* The full Lagrangian describing interaction of Yang-Mills fields with charged fermions is then

$$\mathcal{L} \ = \ \bar{\Psi}(i\gamma^m D_m - M)\Psi - \frac{1}{4} F^a_{mn} F^{a,mn} + \mathcal{L}_\xi + \mathcal{L}_{\text{ghosts}} \ = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \ . \tag{139}$$

In (139), $\mathcal{L}_\xi$ is a gauge fixing term, whereas $\mathcal{L}_{\text{ghosts}}$ is the Fadeev-Popov ghost lagrangian [16], coming from the covariant quantization of non-abelian theories.

Homework : show that for an $SU(2)$ doublet

$$\bar{\Psi}(i\gamma^m D_m - M)\Psi = \bar{\Psi}^k[\delta_{kl}(i\gamma^m \partial_m - M) + \frac{g}{2}\gamma^m W^a_m (\tau_a)_{kl}]\Psi^l \ , \tag{140}$$

whereas the fermion and Yang-Mills field equations are

$$\begin{aligned}(i\gamma^m D_m - M) \, \Psi \ &= \ 0 \ , \\ \partial^m F^a_{mn} + g\epsilon_{abc} A^{b,m} F^c_{mn} \ &= \ - \, g \, \bar{\Psi}\gamma_n \frac{\tau_a}{2} \Psi\end{aligned} \tag{141}$$

where on the right-hand side one can identify the $SU(2)$ fermionic current j^a_n, which can also be constructed according to the Noether procedure. Notice that the massive Yang-Mills field propagator is

$$D^{ab}_{mn}(k) \ = \ -i\delta^{ab} \frac{g_{mn} - \frac{k_m k_n}{M^2_A}}{k^2 - M^2_A} \ . \tag{142}$$

Since here $\partial^m j^a_m \neq 0$, the longitudinal polarization *does contribute* to scattering amplitudes. Therefore, unlike the abelian case, here the UV properties of the massless and massive YM theories are different. This fact has various consequences :

- the theory has bad UV behaviour (uncontrolled UV divergences), since in the UV the propagator behaves as $D^{ab}_{mn}(k) \sim 1/M^2_A$.
- the amplitude $W_L W_L \to W_L W_L$, where W_L is the longitudinal component of the W gauge boson, grows with energy and invalides perturbation theory for energies above around 1.2 TeV.

The conclusion of all these problems is that the Yang-Mills boson masses *should not be added by hand*, but be generated in a more subtle way. On the other hand, massless gauge fields (infinite range) cannot describe electroweak interactions, which are short range. We need therefore to give gauge bosons a mass, but we need *another way* to generate gauge boson masses. This is explained via the spontaneous symmetry breaking and the Higgs mechanism, to which we now turn.

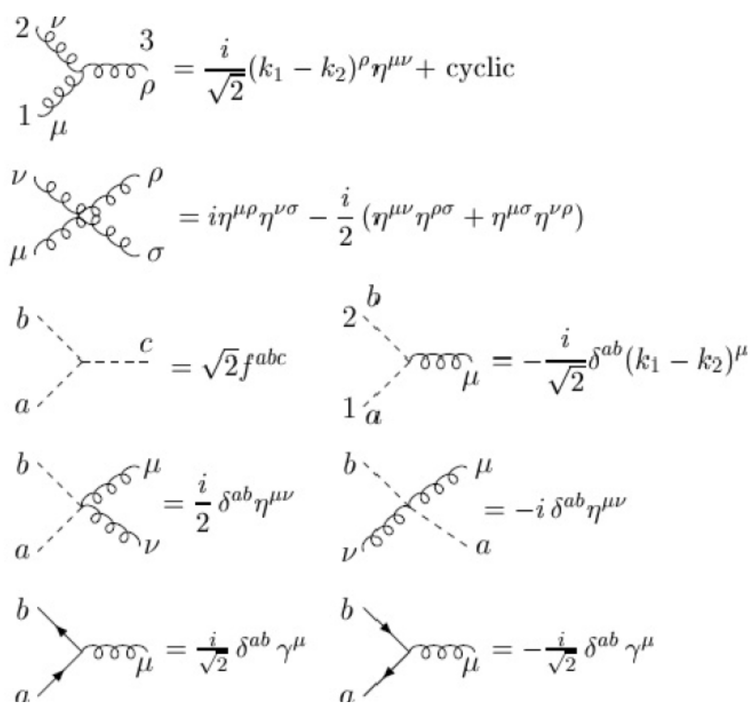


Fig. 5: Feynman rules for Yang-Mills theories. Solid lines are fermion propagators, wavy lines are gauge propagators, whereas the dotted ones are scalars [12]. The structure constants are defined for an arbitrary gauge group via $[T^a, T^b] = i f^{abc} T^c$.

5 Spontaneous symmetry breaking.

We already noticed that *Symmetries*, through the Noether theorem, imply the existence of conserved charges Q^a, commuting with the hamiltonian of the system $[H, Q^a] = 0$. Let us define in what follows the group element implementing symmetry transformations of parameters θ_a, $U(\theta) = e^{i\theta_a Q^a}$. There are two qualitatively different ways symmetries are realized in nature :
i) **Weyl-Wigner (WW) realization** : in this case the vacuum state $|0\rangle$ is *invariant* under the symmetry

$$U(\theta)|0\rangle \equiv e^{i\theta_a Q^a}|0\rangle = |0\rangle \ \Rightarrow \ Q^a|0\rangle = 0 \ . \tag{143}$$

In this case the *symmetry is manifest* in the spectrum and the interactions.
The argument goes as follows: The fields Φ_i of the theory transform according to irreducible representations of the group generated by Q^a

$$U(\theta)\Phi_i U(\theta)^{-1} \ = \ U_{ij}(\theta)\Phi_j \ . \tag{144}$$

Let us consider a quantum state $|i\rangle = \Phi_i|0\rangle$. The action of the symmetry transformation on this state is

$$U(\theta)|i\rangle = U(\theta)\Phi_i U(\theta)^{-1}U(\theta)|0\rangle = U_{ij}(\theta)\Phi_j|0\rangle = U_{ij}(\theta)|j\rangle \ . \tag{145}$$

So the spectrum of the theory is classified in multiplets of the symmetry group. Moreover, since $[H, U(\theta)] = 0$, states in the same multiplet have the same energy.
Simple examples of this type are : translations (conserved charge: momentum), rotations (conserved charge: angular momentum), $U(1)_{em}$ (conserved charge: electric charge)...
ii) **Nambu-Goldstone (NG) realization** : in this case the vacuum state is *not invariant* under the symmetry.

$$e^{i\theta_a Q^a}|0\rangle \neq |0\rangle \ \Rightarrow \ Q^a|0\rangle \neq 0 \ . \tag{146}$$

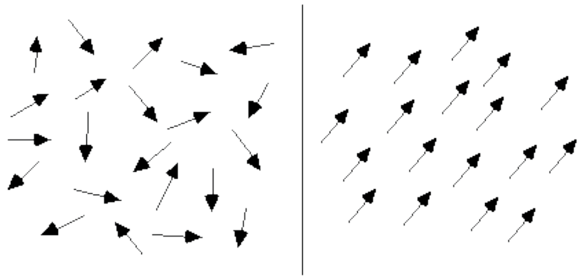


Fig. 6: Magnetization in ferromagnets. At high temperatures, spins orientation are random. At low-temperatures, the alignement of spins due to spin-spin interactions breaks the rotational symmetry of the hamiltonian.

In this case the *symmetry is not manifest* in the spectrum. We talk about spontaneous symmetry breaking. Examples of this type include rotation (or parity) symmetry in ferromagnets, $SU(2)_{weak}$, $SU(2)_L \times SU(2)_R$ chiral symmetry of strong interactions, etc
A nice sentence summarizing the outcome of the two realizations of global symmetries is that of S. Coleman in his Erice Lectures [21] *: "the symmetry of the vacuum is the symmetry of the world"*.
The simplest example of the NG realization is the Ising model describing N spins in space dimension d, of hamiltonian

$$H \; = \; - \, J \sum_{(i,j)} S_i S_j \; - \; B \sum_i S_i \; , \tag{147}$$

with $S_i = \pm 1$ labelling the two possible values of the spin "i". For zero magnetic field $B = 0$ the system has a Z_2 symmetry which reverts the spins $S_i \to -S_i$. As a consequence, the magnetization defined as

$$M = \lim_{B \to 0, N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \langle S_k \rangle$$

should therefore vanish. However experimentally it is known that

$$M = 0 \;\; \text{for} \;\; T \; \geq \; T_c \quad , \quad M \neq 0 \;\; \text{for} \;\; T < T_c, \;\text{where}\; kT_c = 2dJ \; . \tag{148}$$

The reason for the violation of the Z_2 symmetry is that at low temperatures, due to the spin-spin interactions of strength J, spins tend to align, such that the ground state correspond to a state with all spins aligned. This state does violate the Z_2 symmetry, since the Z_2 transform of this ground state is the state with all spins reversed. Whereas both states (vacua) are equally possible, the transition from one to the other is highly suppressed for large N. So if the system is in one of the two vacua, it will stay there a time that scales as e^N. On the other hand, at high-temperature, spins are oriented arbitrarily in order to increase the entropy (number), which wins over the higher-energy of such configurations. This phenomenon is called spontaneous symmetry breaking, since the hamiltonian of the system respects the Z_2 symmetry, which is broken only by the ground state for $T < T_c$. The field theory analog of this phenomenon is described in the next paragraph.

5.1 The Goldstone theorem

In a theory with continuous symmetry, for every generator which does not annihilate the vacuum $\langle T^a \Phi \rangle \neq 0$ *there is a massless, NG particle* [17].
Ex: One of the simplest examples is the $O(N)$ *linear sigma model*.
Consider a theory with N scalar fields $\mathbf{\Phi} = (\Phi_1, \Phi_2, \cdots \Phi_N)$, with lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_m \mathbf{\Phi})^2 - V(\mathbf{\Phi}) \; , \; V(\mathbf{\Phi}) = -\frac{\mu^2}{2} \mathbf{\Phi}^2 + \frac{\lambda}{4} (\mathbf{\Phi}^2)^2 \; , \tag{149}$$

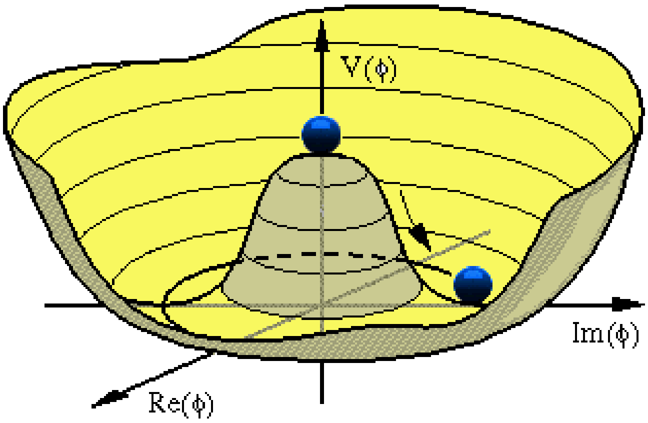


Fig. 7: Spontaneous symmetry breaking. The vacuum manifold minima is symmetric by rotations, but picking up a vacuum breaks the symmetry spontaneously. Figure taken from [18].

where in our convention $\mu^2 > 0$ and where $\mathbf{\Phi}^2 = \sum_{i=1}^{N} \Phi_i \Phi_i$. The model has a continuous $O(N)$ symmetry acting as $\mathbf{\Phi} \to R\mathbf{\Phi}$, with R a $N \times N$ rotation matrix. The scalar potential is minimized for

$$\frac{\partial V}{\partial \Phi_i} = 0 \quad \Rightarrow \quad \mathbf{\Phi_0}^2 = \frac{\mu^2}{\lambda} \equiv v^2 \,. \tag{150}$$

The vacuum manifold is $O(N)$ invariant. By an $O(N)$ rotation, the ground state can be chosen to be

$$\langle \mathbf{\Phi} \rangle \;=\; \mathbf{\Phi_0} \;=\; (0, 0 \cdots v) \,, \tag{151}$$

preserving an $O(N-1)$ subgroup. Goldstone's theorem tells us that we expect the model to have $N-1$ *massless particles*, corresponding to the number of broken generators of the coset group $O(N)/O(N-1)$.

In order to check this, we define a set of shifted fields:

$$\mathbf{\Phi}(x) = (\pi^k(x), v + \sigma(x)) \,, \; k = 1, 2 \cdots, N-1 \,, \tag{152}$$

such that $\langle \pi^k \rangle = \langle \sigma \rangle = 0$. The lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}((\partial_m \pi)^2 + (\partial_m \sigma)^2) - \mu^2 \sigma^2 - \sqrt{\lambda}\, \mu \sigma^3 \\ &- \sqrt{\lambda}\, \mu \pi^2 \sigma - \frac{\lambda}{4}(\sigma^2 + \pi^2)^2 \,, \end{aligned} \tag{153}$$

where $\pi^2 = \sum_{k=1}^{N-1} \pi^k \pi^k$. The manifest symmetry is indeed $O(N-1)$, which rotates the "pions" π's among themselves. The physical masses, visible from (153) are

$$m_\sigma^2 = 2\mu^2 \,, \; m_{\pi_k}^2 = 0 \,. \tag{154}$$

Therefore we find that *the "pions" are massless*; they are the $N-1$ Nambu-Goldstone (NG) bosons of the broken symmetry. It is said that the unbroken $O(N-1)$ symmetry is realized a la Weyl-Wigner, whereas the original $O(N)$ symmetry is realized a la Nambu-Goldstone.

The presence of a continuous symmetry of the lagrangian implies, via the Noether theorem, the existence of a conserved current $\partial_m J^m = 0$. This implies

$$0 = \int d^3\mathbf{x}\, [\partial^m J_m(\mathbf{x}, t), \mathbf{\Phi}(0)] = \partial^0 \int d^3\mathbf{x}\, [J_0(\mathbf{x}, t), \mathbf{\Phi}(0)] + \int d\mathbf{S}\, [\mathbf{J}(\mathbf{x}, t), \mathbf{\Phi}(0)] \,, \tag{155}$$

where the last term is a surface integral. For a sufficiently large surface and therefore space-like distance, the last term vanishes and one obtains

$$\frac{d}{dt}\,[Q(t), \mathbf{\Phi}(0)] \;=\; 0\;, \tag{156}$$

where $Q(t) = \int d^3\mathbf{x}\; J_0(x)$ is the charge associated to the current J_m. If the vacuum expectation value

$$\langle 0|[Q(t), \mathbf{\Phi}(0)]\rangle 0 \equiv \mathbf{v} \neq 0\;, \tag{157}$$

then the symmetry is spontaneously broken. By inserting a complete set of states $|\mathbf{n}\rangle$ and by using the translation operator, one can derive from (157) the relation

$$\sum_{\mathbf{n}} (2\pi)^3\delta^3(\mathbf{p_n})\;\{\langle 0|J_0(0)\rangle\mathbf{n}\langle\mathbf{n}|\mathbf{\Phi}_0(0)\rangle 0\; e^{-iE_{\mathbf{n}}t} - \langle 0|\mathbf{\Phi}_0(0)\rangle\mathbf{n}\langle\mathbf{n}|J_0(0)\rangle 0\; e^{iE_{\mathbf{n}}t}\} \;=\; \mathbf{v}\;. \tag{158}$$

The left-hand side of (158) is non-vanishing and time-independent, according to (156). Since the positive and negative frequencies cannot compensate each other, (158) can hold if and only if all matrix elements are zero for all states except one $|n\rangle$, for which $E_{\mathbf{n}} = 0$ for $\mathbf{p_n} = 0$, i.e. for a *massless state* in the spectrum. This is precisely the Goldstone boson, which has the properties

$$\langle\mathbf{n}|\mathbf{\Phi}_0(0)|0\rangle = 1 \quad , \quad \langle 0|J_m(0)|\mathbf{n}(p)\rangle = i\mathbf{v}p_m\;. \tag{159}$$

Another hint of the existence of a massless particle in this case comes from the relation (74). If the current is conserved, then $\Delta = 0$, however if it is spontaneously broken, then $\delta\langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \neq 0$. Then

$$\lim_{k\to 0} k_m \int d^4z\; e^{-ikz}\langle 0|TJ_m(z)\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle = \delta\langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \neq\; 0\;. \tag{160}$$

This is possible only if the correlation function has a massless pole

$$\lim_{k\to 0}\int d^4z\; e^{-ikz}\langle 0|TJ_m(z)\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \sim \frac{-k_m}{k^2}\;\delta\;\langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle\;, \tag{161}$$

which implies the existence of a massless Goldstone boson oupled to the current J_m, in agreement with (159).

General (classical) proof of the Goldstone theorem.

Consider the scalar theory of lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_m\Phi_i)^2 - V(\Phi_i) \tag{162}$$

and a *global continuous symmetry group*, of generators T^a. The invariance of the scalar potential

$$V(\Phi_i + \delta\Phi_i) \;=\; V(\Phi_i) \tag{163}$$

under infinitesimal transformations $\delta\Phi_i \;=\; i\theta^a T^a_{ij}\Phi_j$ of parameters θ^a implies

$$\frac{\partial V}{\partial\Phi_i}T^a_{ij}\Phi_j \;=\; 0\;. \tag{164}$$

Differentiating again and taking the vacuum expectation value (vev), we get

$$\langle\frac{\partial^2 V}{\partial\Phi_k\partial\Phi_i}T^a_{ij}\Phi_j + \frac{\partial V}{\partial\Phi_i}T^a_{ik}\rangle = 0\;. \tag{165}$$

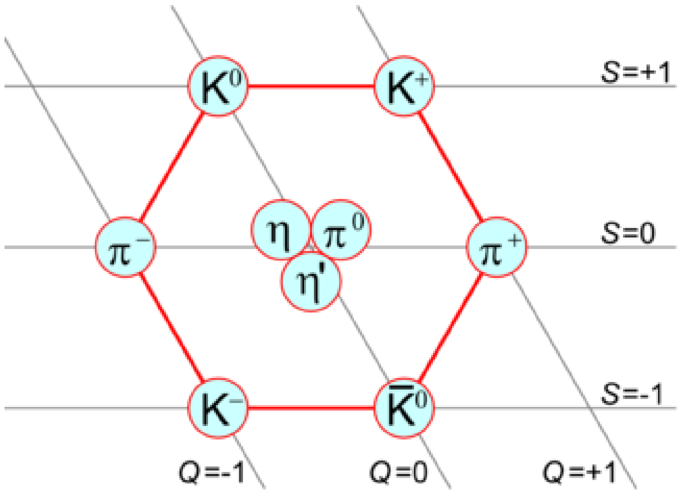


Fig. 8: The meson octet acts as goldstone bosons of the chiral symmetry breaking $SU(3)_L \times SU(3)_R \to SU(3)_V$.Figure taken from [19].

Remembering that $\mathcal{M}^2_{ki} = \langle \frac{\partial^2 V}{\partial \Phi_k \partial \Phi_i} \rangle$ is the scalar mass matrix, we obtain

$$\mathcal{M}^2_{ki}\, (T^a v)_i \ = \ 0\ . \tag{166}$$

We therefore found the general form of the **Goldstone theorem**: *If the vacuum is not invariant under a symmetry generator $T^a v \neq 0$, then $T^a v$ is an eigenvector of the mass matrix $\mathcal{M}^2$ corresponding to a zero eigenvalue.*

Are there known examples of Goldstone bosons in nature ? Yes, there are several, but none of them *not corresponding to a fundamental spin* 0 *particle !*. Two well-known examples are
- Magnons spin waves in ferromagnets, which are long wavelength collective spin configurations.
- Pions $\pi \sim q\ \bar{q}$ are *pseudo-Goldstones* for the breaking of the *chiral* $\to$ vector symmetries $U(3)_L \times U(3)_R \to SU(3)_V \times U(1)_B$ (see figure 8). They are not exactly massless (therefore the name "pseudo") due to a small explicit breaking coming from quark masses. Pions are (pseudo)scalar particles, but not elementary, they are quark-antiquark bound states. In this case we talk about *dynamical* symmetry breaking.

Observation : The $U(1)_A$ symmetry is broken by quantum anomalies, there is no corresponding goldstone boson.

5.2 Chiral symmetries and pions as goldstone bosons

Strong interactions have as elementary degrees of freedom quarks and gluons and are described at the microscopic (UV) level by Quantum Chromodynamics (QCD). At high energy QCD is perturbative due to asymptotic freedom and its predictions agree remarquably well with the high-energy data. At low-energy, it becomes nonperturbative and it is very difficult to perform ab-initio QCD computations to describe properties and interactions of hadrons. At low-energy, the mesons and baryons are the relevant degrees of freedom; they interactions, although hard to derive directly from QCD, are constrained from symmetries of strong interactions. The model described below is an excellent first step towards a phenomenological description of strong interactions between pions and the nucleons. The corresponding effective lagrangian is called the linear sigma model, defined by

$$\mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_1\ , \tag{167}$$

where :

$$\begin{aligned}
\mathcal{L}_0 &\equiv \overline{\psi}_a(x)\left[i\delta_{ab}\partial\!\!\!/ - g(\sigma(x)\delta_{ab} + i\pi^i(x)\tau^i_{ab}\gamma_5)\right]\psi_b(x) + \frac{1}{2}(\partial_\mu\sigma(x)\partial^\mu\sigma(x) + \partial_\mu\pi^i(x)\partial^\mu\pi^i(x)) \\
&\quad - \frac{\mu^2}{2}(\sigma^2(x) + \pi^i(x)\pi^i(x)) - \frac{\lambda}{4}(\sigma^2(x) + \pi^i(x)\pi^i(x))^2 , \\
\mathcal{L}_1 &\equiv c\,\sigma(x) . \qquad (168)
\end{aligned}$$

The spinors $\psi_a(x)$ ($a = 1, 2$) are fermion doublets, the proton and the neutron [3], $\sigma(x)$ et $\pi^i(x)$ ($i = 1, 2, 3$) are scalar fields representing respectively the sigma[4] and the three pions. We take λ and ϵc to be positive, but μ^2 can be positive or negative. The matrices τ^i ($i = 1, 2, 3$) are the three Pauli matrices.

In order to avoid a too heavy notation, the index a of the fermion doublets and the internal index i will be omitted by using the following notations:

$$\boldsymbol{\psi}(x) \equiv \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} , \quad \boldsymbol{\psi}^\dagger(x) \equiv (\psi_1^\dagger(x)\ \psi_2^\dagger(x)) , \quad \boldsymbol{\pi}(x) \equiv \begin{pmatrix} \pi^1(x) \\ \pi^2(x) \\ \pi^3(x) \end{pmatrix} , \quad \boldsymbol{\tau} \equiv \begin{pmatrix} \tau^1 \\ \tau^2 \\ \tau^3 \end{pmatrix} . \qquad (169)$$

The Lagrangian $\mathcal{L}_0$ can therefore be written in the compact form

$$\mathcal{L}_0 = \overline{\boldsymbol{\psi}}[i\partial\!\!\!/ - g(\sigma + i\boldsymbol{\pi}\cdot\boldsymbol{\tau}\gamma_5)]\boldsymbol{\psi} + \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}) - \frac{\mu^2}{2}(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2)^2 \qquad (170)$$

In order to distinguish the left and right fermion chiralities, it is necessary to introduce the chiral projectors :

$$P_R \equiv \frac{1+\gamma_5}{2} , \qquad P_L \equiv \frac{1-\gamma_5}{2} . \qquad (171)$$

Using them, we define chiral spinors:

$$\psi_{L,R} \equiv P_{L,R}\psi , \qquad \overline{\psi}_{L,R} = \overline{\psi}P_{R,L} . \qquad (172)$$

It can be shown that the lagrangian is invariant under a global symmetry group $SU(2)\times SU(2)$. In order to check this, let U_R and U_L be two global $SU(2)$ matrices. Then, $\mathcal{L}_0$ is invariant under the following transformation:

$$\begin{aligned}
&\boldsymbol{\psi}_R \to U_R\boldsymbol{\psi}_R \quad , \quad \boldsymbol{\psi}_L \to U_L\boldsymbol{\psi}_L , \\
&M \equiv \sigma I_{2\times 2} + i\boldsymbol{\pi}\cdot\boldsymbol{\tau} \to U_L\, M\, U_R^\dagger . \qquad (173)
\end{aligned}$$

Indeed, this can be easily verified writing the lagrangian in the more elegant and transparent form

$$\begin{aligned}
\mathcal{L}_0 &= \overline{\boldsymbol{\psi}}_L i\partial\!\!\!/\boldsymbol{\psi}_L + \overline{\boldsymbol{\psi}}_R i\partial\!\!\!/\boldsymbol{\psi}_R - g(\overline{\boldsymbol{\psi}}_L M\boldsymbol{\psi}_R + \overline{\boldsymbol{\psi}}_R M^\dagger\boldsymbol{\psi}_L) \\
&+ \frac{1}{4}Tr(\partial_\mu M^\dagger\partial^\mu M) - \frac{\mu^2}{4}Tr(M^\dagger M) - \frac{\lambda}{16}(Tr M^\dagger M)^2 . \qquad (174)
\end{aligned}$$

Notice that the term $\mathcal{L}_1$ is only invariant under the transformation vector transformations defined by $U_L = U_R$. The infinitesimal form of these transformations can be found starting from the parametrization

$$U_R = e^{\frac{1}{2}(\alpha-\beta)\tau} , \; U_L = e^{\frac{1}{2}(\alpha+\beta)\tau} , \qquad (175)$$

where α (β)) parametrize the vector (axial) transformations. Then the infinitesimal global transformations are

$$\sigma \to \sigma - \boldsymbol{\beta}\cdot\boldsymbol{\pi} \quad , \quad \boldsymbol{\pi} \to \boldsymbol{\pi} - \boldsymbol{\alpha}\times\boldsymbol{\pi} + \boldsymbol{\beta}\sigma ,$$

[3] In the "quark constituants model", the doublet spinors are the quarks u and d instead of the proton and neutron

[4] This particle appears naturally in the effective theories describing interactions between pions and nucleons if we require an invariance under an axial symmetry. To date, it is not identified unambiguously in the experimental data

$$
\begin{aligned}
\boldsymbol{\psi}_{_L} &\to \boldsymbol{\psi}_{_L} + \frac{i}{2}(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{\tau}\boldsymbol{\psi}_{_L}\ , \\
\boldsymbol{\psi}_{_R} &\to \boldsymbol{\psi}_{_R} + \frac{i}{2}(\boldsymbol{\alpha}-\boldsymbol{\beta})\cdot\boldsymbol{\tau}\boldsymbol{\psi}_{_R}\ ,
\end{aligned} \tag{176}
$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are infinitesimal vectors. Notice that $\mathcal{L}_1$ is only invariant under the vector transformations induced by $\boldsymbol{\alpha}$. The symetry induced by the vector $\boldsymbol{\alpha}$ is called "vector symmetry", whereas that correspondent to the vector $\boldsymbol{\beta}$ is called "axial symmetry".

Let us denote $V^i_\mu(x)$ (vector current) the current associated to the infinitesimal transformations parameterized by the vector $\boldsymbol{\alpha}^i$ and $A^i_\mu(x)$ (axial current) the current associated to the transformations parameterized by the vector $\boldsymbol{\beta}^i$. They are given explicitly by

$$
\begin{aligned}
\mathbf{V}_\mu &= \bar{\psi}\gamma^\mu\frac{\boldsymbol{\tau}}{2}\psi + \boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}\ , \\
\mathbf{A}_\mu &= \bar{\psi}\gamma^\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi + \sigma\partial_\mu\boldsymbol{\pi} - \boldsymbol{\pi}\partial_\mu\sigma\ .
\end{aligned} \tag{177}
$$

By using Noether theorem, we find the conservation laws

$$
\partial^\mu V^i_\mu(x)\ =\ 0\ , \qquad \partial^\mu A^i_\mu(x)\ =\ -\epsilon\, c\, \pi^i(x)\ . \tag{178}
$$

Therefore the vector symmetry is conserved, whereas the lagrangian $\mathcal{L}_1$ describes the explicit breaking of the axial symmetry.

One can show that the vector and axial currents satisfy some group commutation relations, called *current algebra*. By using the canonical commutation and anticommutation relations at equal times

$$
\begin{aligned}
&\delta(x^0-y^0)\left[\pi^i(x),\partial^0\pi^j(y)\right] = i\delta^4(x-y)\delta^{ij}\ , \\
&\delta(x^0-y^0)\left[\sigma(x),\partial^0\sigma(y)\right] = i\delta^4(x-y)\ , \\
&\delta(x^0-y^0)\left\{\psi_{\alpha a}(x),\psi^\dagger_{\beta b}(y)\right\} = \delta^4(x-y)\delta_{\alpha\beta}\delta_{ab}
\end{aligned} \tag{179}
$$

it can be shown that

$$
\begin{aligned}
&\delta(x^0-y^0)\left[V^i_0(x),V^j_\mu(y)\right] = i\delta^4(x-y)\epsilon^{ijk}V^k_\mu(x) + \text{S.T.}\ , \\
&\delta(x^0-y^0)\left[V^i_0(x),A^j_\mu(y)\right] = i\delta^4(x-y)\epsilon^{ijk}A^k_\mu(x) + \text{S.T.}\ , \\
&\delta(x^0-y^0)\left[A^i_0(x),A^j_\mu(y)\right] = i\delta^4(x-y)\epsilon^{ijk}V^k_\mu(x) + \text{S.T.}\ ,
\end{aligned} \tag{180}
$$

where the supplementary terms denoted S.T (from "Schwinger terms") are only present when $\mu \neq 0$ and are total derivatives of local distributions:

$$
\boldsymbol{\nabla}_y\left[f(x,y)\delta^4(x-y)\right]\ . \tag{181}
$$

The relations (180) form the so-called Gell-Mann current algebra and play an important role in the study of low-energy interactions of pions and nucleons. One can also define the charges:

$$
Q^i_\pm(x^0) \equiv \frac{1}{2}\int d^3\boldsymbol{x}(V^i_0(x) \pm A^i_0(x))\ . \tag{182}
$$

Using the commutation relations previously established, one can prove the commutation relations

$$
\left[Q^i_\pm, Q^j_\pm\right] = i\epsilon^{ijk}Q^k_\pm\ , \qquad \left[Q^i_+, Q^j_-\right] = 0\ . \tag{183}
$$

These relations are precisely those of the Lie algebra $SU(2)\times SU(2)$, implementing the chiral symmetry of the system.

The scalar potential of the linear sigma model is given by

$$V = \frac{\mu^2}{2}(\sigma^2 + \boldsymbol{\pi}^2) + \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2)^2 - \epsilon c\,\sigma\ . \tag{184}$$

Before discussing the minimization of the scalar potential, let us first state our expectations based on symmetry arguments and Goldstone theorem. For $\epsilon = 0$, the scalar potential and actually the whole lagrangian has the symmetry $SO(4) = SU(2)_L \times SU(2)_R$. One can treat $(\boldsymbol{\pi}, \sigma)$ as a vector under $SO(4)$. We can always use a rotation to choose the vacuum of the system of the form $(0, \sigma_0 = v)$, which preserves only a $SO(3) = SU(2)_V$ subgroup of the original symmetry. This is also easily seen by using the matrix M, in which case $\langle M \rangle = vI_{2\times 2}$, which is invariant under the subgroup $U_L = U_R$. We therefore expect three goldstone bosons related to the spontaneous breaking of the axial symmetries generated by the parameters $\boldsymbol{\beta}$, which will be readily identified with the pions $\boldsymbol{\pi}$. The symmetry breaking term $\mathcal{L}_1$ breaks explicitly the axial symmetries and therefore we expect to give a mass to the pions. The minimization of the potential can be safely done for $\boldsymbol{\pi} = 0$. The minimum σ_0 of the potential is determined by the equation

$$(\mu^2 + \lambda\sigma_0^2)\,\sigma_0 = \epsilon c\ . \tag{185}$$

The minima of the potential are qualitatively different in the case $\mu^2 > 0$ and $\mu^2 < 0$. Indeed, for $\mu^2 > 0$ it can easily ne checked graphically that there is only one minimum, whereas for $\mu^2 < 0$ there are two minima. More interesting, when $\epsilon = 0$ and $\mu^2 < 0$, the minima are non-symmetric, even if the Lagrangian is invariant under the axial transformations. We then talk about *spontaneous* symmetry breaking of the axial symmetries, a consequence of the general Goldstone theorem we disscused in the previous paragraph.

In order to work out the physical spectrum, one defines $\sigma(x) = s(x) + \sigma_0$ where σ_0 is the constant field σ in the ground state.

We are primarily interested in the limit $\epsilon\, c \to 0$, where the full Lagrangian is invariant under axial transformations and

– if $\mu^2 > 0$, we have the symmetric case:

$$\sigma_0 = 0\ , \quad m_\psi = 0 \quad , \quad m_\pi^2 = m_\sigma^2 = \mu^2. \tag{186}$$

– if $\mu^2 < 0$, the symmetry of the Lagrangian is spontaneously broken in the ground state and

$$\sigma_0^2 = v^2 = -\mu^2/\lambda\ , \quad m_\psi = gv\ , \quad m_\sigma^2 = 2\lambda v^2 = -2\mu^2\ , \quad m_\pi = 0\ . \tag{187}$$

The pions are therefore indeed massless, in agreement with the Goldstone theorem. For $\epsilon \neq 0$, it can be readily checked that the pions get a mass from the explicit axil symmetries breaking. For small ϵ, pions mass is given by $m_\pi = \epsilon v\sqrt{\frac{\lambda}{-\mu^2}}$.

A natural question arises: *What happens if the spontaneously broken symmetry is gauged ?* The answer is given in the next subsection.

5.3 The Higgs mechanism

Let us start for simplicity with an *abelian gauge theory*

$$\mathcal{L} = -\frac{1}{4}F_{mn}^2 + |D_m\Phi|^2 - V(\Phi)\ , \tag{188}$$

with $D_m = \partial_m + ieA_m$, $\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$ and a scalar potential

$$V = -\mu^2|\Phi|^2 + \lambda(|\Phi|^2)^2 = -\frac{\mu^2}{2}(\Phi_1^2 + \Phi_2^2) + \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2)^2\ , \tag{189}$$

invariant under the *local* $U(1)$ gauge transformations

$$\Phi \to e^{i\alpha(x)}\Phi \ , \ A_m \to A_m - \frac{1}{e}\partial_m \alpha \ . \tag{190}$$

We expand around the vacuum state

$$\Phi_0 = \sqrt{\frac{\mu^2}{2\lambda}} = \frac{v}{\sqrt{2}} \ , \ \Phi(x) = \frac{1}{\sqrt{2}}(v + \phi_1 + i\phi_2) \ . \tag{191}$$

From the quadratic mass terms in the scalar potential we find $m_1^2 = 2\mu^2$, $m_2 = 0$, therefore ϕ_2 is the Goldstone boson. New features appear however from the kinetic term

$$|D_m\Phi|^2 = \frac{1}{2}(\partial_m\phi_i)^2 + evA_m\partial^m\phi_2 + \frac{e^2v^2}{2}A_m^2 + \cdots \tag{192}$$

Indeed, it is manifest from (192) that *the gauge boson acquired a mass* $M_A^2 = e^2v^2$. But this can only happen if the gauge field absorbed one degree of freedom, since the massive gauge field has three degrees of freedom, whereas the massless one has only two degrees of freedom. The correct counting of degrees of freedom is

$$A_m(M_A = 0) + \phi_2 \ \to \ A_m(M_A \neq 0) \tag{193}$$

That this is indeed true can be seen in various ways:
i) The quadratic term in the lagrangian can be diagonalized by redefining the gauge field

$$\begin{aligned} & -\frac{1}{4}F_{mn}^2 + \frac{1}{2}(\partial_m\phi_2)^2 + \sqrt{2}evA_m\partial^m\phi_2 + \frac{e^2v^2}{2}A_m^2 \\ & = -\frac{1}{4}(\partial_m B_n - \partial_n B_m)^2 + \frac{e^2v^2}{2}B_m^2 \ , \end{aligned} \tag{194}$$

where $B_m = A_m + \frac{1}{ev}\partial_m\phi_2$. Therefore ϕ_2 *disappeared from the quadratic part, and is "absorbed" into the longitudinal component of the gauge field.*
ii) The Goldstone can be *eliminated altogether* from the lagrangian in the so-called *unitary gauge*. The corresponding parametrization is

$$\Phi(x) = \frac{1}{\sqrt{2}}\, e^{\frac{i\theta(x)}{v}}\, (v + \rho(x)) \tag{195}$$

and the Goldstone is removed by the gauge transformation $\Phi \to \Phi' = e^{-\frac{i\theta}{v}}\Phi$, $A_m \to A'_m = A_m + \frac{1}{ev}\partial_m\theta$.

In the unitary gauge, the lagrangian is (**homework**)

$$\mathcal{L} = -\frac{1}{4}(F'_{mn})^2 + (\partial_m - ieA'_m)\Phi'(\partial^m + ieA'^m)\Phi' - \mu^2\Phi'^2 - \lambda\Phi'^4$$

The spectrum of the model contains therefore a massive gauge boson and *the Higgs boson* Φ'*, of mass* $2\mu^2$ [1].

The Higgs mechanism, non-abelian case

Consider a gauge group G of rank r and scalar fields in some irreducible n-dim. representation

$$\mathcal{L} = -\frac{1}{4}F^a_{mn}F^{a,mn} + |[(\partial_m - igT^aA^a_m)\Phi]|^2 - V(\Phi) \ , \tag{196}$$

and $H \in G$ the subgroup of rank s leaving the ground state invariant

$$T^a v = 0 \quad , \quad a = 1 \cdots s$$

$$T^a v \neq 0 \quad , \quad a = s+1 \cdots r \tag{197}$$

In the unitary gauge parametrization

$$\Phi(x) = e^{i \sum_{a=s+1}^{r} T_a \frac{\xi_a(x)}{v}} \; \frac{\rho(x)+v}{\sqrt{2}} \;, \tag{198}$$

where ξ_a are the Goldstone bosons, $\rho(x)$ the remaining scalar fields, and $\langle \xi_a \rangle = \langle \rho \rangle = 0$. The gauge transformation

$$\begin{aligned} &\Phi(x) \to \Phi'(x) = U\Phi \;, \text{ with } U = e^{-i \sum_{a=s+1}^{r} T_a \frac{\xi_a(x)}{v}} \\ &A_m \to A'_m = U \, (A_m + \frac{i}{g}\partial_m) \, U^{-1} \end{aligned} \tag{199}$$

eliminates the Goldstone bosons from the lagrangian. The resulting mass matrix of the vector fields is then

$$M^2_{ab} = g^2 (T_a v)^\dagger (T_b v) = \frac{g^2}{2} v^\dagger \{T_a, T_b\} v \;. \tag{200}$$

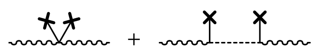
In this case $r-s$ gauge bosons become massive

$$A^a_m + \xi_a \;\to\; A'^a_m = A^a_m - \frac{1}{v} D_m \xi^a + \cdots \tag{201}$$

where A_m denote the massless gauge fields, containing two degrees of freedom, whereas A'_m denote the massive gauge fields.
Notice that *the number of physical massive Higgs scalars is equal to the number of original scalars, minus the number of broken gauge generators.*

- Gauge boson propagator

Let us discuss the massive gauge boson propagator in the case of the abelian case; the result being the same in the non-abelian case, the conclusion will holds in both cases.
At tree-level, the photon polarization tensor is given by

$i\Pi_{\mu\nu}(k) =$  , (202)

which equals [5]:

$$i\Pi_{\mu\nu}(k) = ig^2 v^2 g_{\mu\nu} + (-gv)^2 k_\mu (-k_\nu) \frac{i}{k^2+i\epsilon} = ig^2 v^2 \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 + i\epsilon} \right] \;. \tag{203}$$

The free photon propagator in the Feynman gauge [6] is given by

$$D_0^{\mu\nu}(k) = -\frac{i g^{\mu\nu}}{k^2 + i\epsilon} \;. \tag{204}$$

We call $D^{\mu\nu}(k)$ the photon propagator resulting from the summation of $\Pi^{\mu\nu}(k)$ at all orders:

$$D^{\mu\nu}(k) = D_0^{\mu\nu}(k) + D_0^{\mu\alpha}(k)(i\Pi_{\alpha\beta}(k)) D_0^{\beta\nu}(k) + \cdots \tag{205}$$

This is a geometric series which becomes trivial if we notice that

$$\left[g_{\mu\alpha} - \frac{k_\mu k_\alpha}{k^2+i\epsilon} \right] \left[g^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2+i\epsilon} \right] = g_\mu{}^\beta - \frac{k_\mu k^\beta}{k^2+i\epsilon} \;. \tag{206}$$

[5] The momentum of the line attached to $\langle 0|\phi|0\rangle$ is zero since the vev's is spacetime independent.

[6] The polarization tensor we obtained is transverse. It will therefore not modify the gauge-dependent term if we would work in a different gauge.

We obtain:

$$
\begin{aligned}
D^{\mu\nu}(k) &= D_0^{\mu\nu}(k) + D_0^{\mu\alpha}(k)\left[\sum_{n=1}^{+\infty}\left(\frac{g^2v^2}{k^2+i\epsilon}\right)^n\right]\left[g_\alpha{}^\nu - \frac{k_\alpha k^\nu}{k^2+i\epsilon}\right] \\
&= -\frac{i}{k^2-g^2v^2+i\epsilon}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+i\epsilon}\right] - i\frac{k^\mu k^\nu}{(k^2+i\epsilon)^2}\ , \qquad (207)
\end{aligned}
$$

where the first term is the propagator of a massif gauge boson of mass $m_{_A} = gv$, whereas the second term is a longitudinal term depending on the chosen gauge. The longitudinal mode contains now a physical mode, as expected for a massive gauge boson. An important observation is that the massive gauge propagator (207) behaves as $D^{\mu\nu}(k) \sim 1/k^2$ in the UV $k \to \infty$. This has the crucial consequence that in the non-abelian case the theory with spontaneous symmetry breaking and massive gauge boson is *renormalizable*, unlike its naive version with a mass introduced by hand (142)[7].

5.4 Quantization of spontaneously broken gauge theories : R_ξ gauges

- The abelian case

Let us come back to the Higgs mechanism and start with the abelian case. One can notice from (192) the mixing between the gauge field and the would-be Goldstone boson. For the quantization of the theory, it is more convenient to work in a gauge where such a term is absent. From (191) and by defining $\Phi_1 = v + h, \Phi_2 = \varphi$, a convenient gauge fixing term is then

$$
\mathcal{L}_{\text{g.f.}} = -\frac{1}{2\xi}\left(\partial_m A^m - e\xi v\varphi\right)^2\ . \qquad (208)
$$

The quadratic part of the lagrangian, which will determine the propagator, becomes in this class of gauges

$$
\begin{aligned}
\mathcal{L}_2 &= -\frac{1}{2}A_m\left[-\eta^{mn}(\Box + (ev)^2) + (1-\frac{1}{\xi})\partial^m\partial^n\right] \\
&- \frac{1}{2}h(\Box + m_h^2)h\ -\ \frac{1}{2}\varphi(\Box + \xi(ev)^2))\varphi\ , \qquad (209)
\end{aligned}
$$

from which one can deduce the massige gauge field, the higgs and and the goldstone boson propagators

$$
\begin{aligned}
\Delta_{mn}(k) &= \frac{-i}{k^2-M_A^2}\left(\eta_{mn} - (1-\xi)\frac{k_m k_n}{k^2-\xi M_A^2}\right)\ , \\
D_h(k) &= \frac{i}{k^2-m_h^2}\quad ,\quad D_\varphi(k) = \frac{i}{k^2-\xi M_A^2}\ , \qquad (210)
\end{aligned}
$$

where $M_A = ev$ is the mass of the gauge boson. The gauge parameter ξ is unphysical and has to cancel in the computation of all physical processes. Notice also the unphysical pole in the goldstone propagator at $k^2 = \xi M_A^2$.

Unlike the case of QED, the ghost fields cannot be completely neglected. Indeed, following the procedure described in the Appendix, one finds their lagrangian to be

$$
\mathcal{L}_{\text{ghosts}} = \bar{c}\left[-\Box - \xi M_A^2(1+\frac{h}{v})\right]c\ . \qquad (211)
$$

The ghosts couple therefore to the Higgs and this interaction has to be taken into account in quantum computations.

[7] The massive QED with a mass added by hand, called Proca theory, is however renormalizable. Indeed, the part of the gauge propagator with bad UV behaviour, prop. to $k^\mu k^\nu/k^2M^2$, does not contribute to scattering amplitudes, due to the conservation of the electromagnetic current. This property does not hold anymore, however, in non-abelian theories.

The naive massive gauge field propagator (or equivalently, the gauge field propagator in the unitary gauge) is obtained in the limit $\xi \to \infty$. The renormalizability in this limit is clearly subtle and the unitary gauge is not practical for quantum computations.

- The non-abelian case

For any complex scalar field representation χ of a gauge group G, one can define a corresponding real representation ϕ via $\phi^T = \sqrt{2}(\mathrm{Re}\chi, \mathrm{Im}\chi)$. The gauge generators $T_a$in this representation are real and antisymmetric. The starting point lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^a_{mn} F^{mn,a} + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{ghosts}} + \frac{1}{2} (D_m \phi_i)^2 - V(\phi) \ . \tag{212}$$

The gauge covariant derivative and gauge transformations are

$$D_m \phi_i = \partial_m \phi_i + g A^a_m T^a_{ij} \phi_j \quad , \quad \delta \phi_i = -g \alpha_a T^a_{ij} \phi_j \ . \tag{213}$$

One define $\phi_i = v_i + \chi_i$. For finding the masses and the propagators, one expands the lagrangian to the quadratic order. In the absence of the gauge fixing term (and the associated ghost lagrangian), one would find

$$\begin{aligned} \mathcal{L}_2 \ &= \ -\frac{1}{2} A^a_m \left[-\eta^{mn} (\delta_{ab} \Box + M^2_{A,ab}) + \delta_{ab} \partial^m \partial^n \right] A^b_n \\ &+ \frac{1}{2} (\partial_m \chi_i)^2 + g A^a_m (T^a v) \partial^m \chi - \frac{1}{2} \chi_i M^2_{\chi,ij} \chi_j \ , \end{aligned} \tag{214}$$

where $M^2_{\chi,ij} = \frac{\partial^2 V}{\partial \chi_i \partial \chi_j}$. The R_ξ gauge in this case is defined by

$$\mathcal{L}_{\text{g.f.}} \ = \ -\frac{1}{2\xi} \left(\partial^m A^a_m - g \xi \chi T_a v \right)^2 \ . \tag{215}$$

As in the abelian case, the mixing Goldstones-gauge field is removed in this case. Following the procedure in the Appendix, one also finds the ghost lagrangian

$$\mathcal{L}_{\text{ghosts}} \ = \ \bar{c}^a \left[-(\partial_m D^m)_{ab} - \xi g^2 (T_a v)(T_b v + T_b \chi) \right] c^b \ . \tag{216}$$

Notice that the ghosts acquired also a (non-physical, of course) mass matrix. The propagators of gauge fields, scalars and ghosts are given by

$$\begin{aligned} \Delta_{mn}(k) \ &= \ \frac{-i}{k^2 - M_A^2} \left(\eta_{mn} - (1-\xi) \frac{k_m k_n}{k^2 - \xi M_A^2} \right) \ , \\ D_\chi(k) &= \frac{i}{k^2 - \xi g^2 (T_a v)(T_a v) - M^2_\chi} \ , \\ D_g(k) \ &= \ \frac{i}{k^2 - \xi M_A^2} \end{aligned} \tag{217}$$

where all propagators have to be understood as matrices, $(M_A^2)_{ab} = g^2 (T_a v)(T_b v)$ is the mass matrix of the gauge bosons. The mass matrix for the scalars is $\xi g^2 (T_a v)_i (T_a v)_j + (M^2_\chi)_{ij}$; the first term is the (non)physical mass matrix for Goldstone bosons, whereas the second part is the physical mass matrix for the massive Higgs-like scalars. They live actually in orthogonal supspaces. Notice that in the unitary gauge $\xi \to \infty$ the Goldstone's and the ghosts disappear from the spectrum. In the unitary gauge however, the massive gauge field propagator is given by (122), which is badly behaved in the UV, making this gauge unpractical for most quantum computations.

Here are some examples of Higgs boson spectra :

- In the Standard Model with *one* Higgs doublet, there is one physical real Higgs scalar $4 - 3 = 1$, where

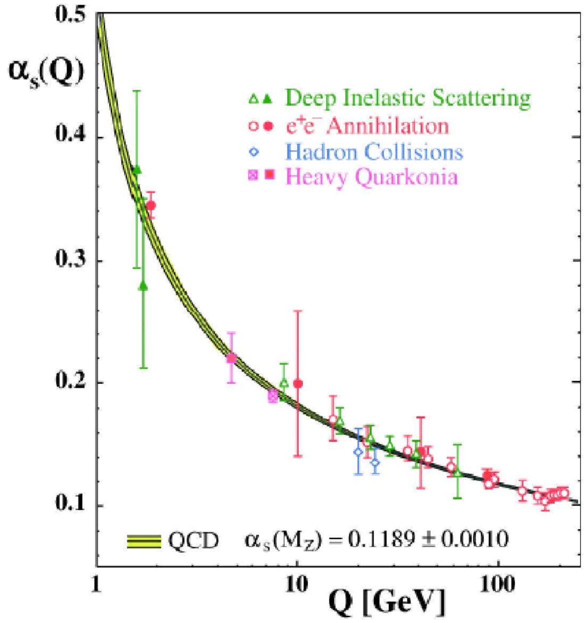


Fig. 9: A confining force similar to QCD called technicolor could be responsible for electroweak symmetry breaking. The electroweak vev is given by a condensate of fermions $\langle \bar{T}_L T_R \rangle \sim v^3 = M_P^3 \, e^{-\frac{3}{2b_0 g_{TC}^2}}$.Taken from [20].

4 is the number of initial real degrees of freedom contained into an $SU(2)_L$ scalar doublet, whereas 3 is the number of broken generators in the Standard Model.
- In the Standard Model with *two* Higgs doublets (or in the Minimal Supersymmetric Standard Model, MSSM), there are $8 - 3 = 5$ physical Higgs scalars: two neutral scalars h and H^0, one pseudoscalar A and two charge ones $H^{\pm}$.
The Higgs mechanism is a elegant and economical way to break electroweak symmetry. *However*, it has its own misteries :
- elementary scalars were never observed in nature until july 2012.
- It is difficult to keep a scalar light after quantum corrections (*so-called hierarchy problem*). We will come back later on to quantify this problem.

Taken into account these observations, it is reasonable to ask is there are other ways of breaking a gauge (electroweak for our purposes) symmetry. The answer is yes, there are several other options. Some popular ones are :
- A new confining force (*technicolor*) with $\Lambda_{TC} \sim v$. The goldstone bosons "eaten up" by the W and Z gauge bosons are called "technipions" (see Fig. 9).
- composite Higgs models, in which Higgs is a bound state of fermions. One example is the top-antitop condensation, where the Higgs is a top-antitop bound-state $h = \bar{t}_L \, t_R$.
- symmetry breaking by boundary conditions in extra-dimensional Kaluza-Klein (string) type theories. Typically these theories have additional light degrees of freedom below the TeV scale; as such, unlike the standard Higgs mechanism, they are strongly constrained and they have problems to fit the experimental data. The recent LHC data on the newly discovered boson, its mass and its couplings point for the time being into the direction of an elementary scalar degrees of freedom like in the Higgs mechanism described above. Consequently, the alternatives we just described to the Higgs mechanism are currently disfavored by the experimental data.

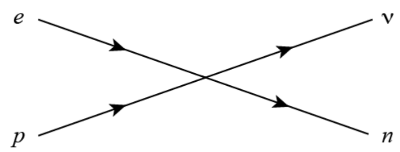


Fig. 10: Fermi theory of weak interactions: the beta decay $n \to pe^-\bar{\nu}_e$ at low energies $E << M_W$ can be described as an effective four-fermion interaction.

6 The electroweak sector of the Standard Model.

6.1 Gauge group and matter content

The *Standard model* is a "unified" description of weak and electromagnetic interactions. From the Fermi theory of weak interactions with $G_F/\sqrt{2} = g^2/8M_w^2$, we know that we need a theory that contains at least a charged gauge boson $W_m^\pm$ and the photon A_m.

Experimentally, there also exists *neutral currents* discovered in 1973, mediated by a neutral massive gauge boson, and also *coloured strong interactions*. The SM gauge group is therefore

Gauge bosons : $\qquad G_m^A \qquad A_m^a \qquad B_m$

$$G \;=\; SU(3)_c \;\times SU(2)_L \;\times U(1)_Y \;.$$

In addition to the gauge bosons, the SM contains matter fermions and the Higgs field, in the gauge group representations

Leptons : $l_i = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L \; : \; (\mathbf{1},\mathbf{2})_{Y=-1} \quad , \quad e_{iR} \; : \; (\mathbf{1},\mathbf{1})_{Y=-2}$

Quarks : $q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L \; : \; (\mathbf{3},\mathbf{2})_{Y=1/3} \; , \; u_{iR} \; : \; (\mathbf{3},\mathbf{1})_{Y=4/3} \; , \qquad d_{iR} \; : \; (\mathbf{3},\mathbf{1})_{Y=-2/3}$

Higgs field : $\Phi = \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix} : \quad (\mathbf{1},\mathbf{2})_{Y=1} \;.$

In the Standard Model, the *Higgs doublet* vev breaks the electroweak gauge sector down to the electric charge $SU(2)_L \times U(1)_Y \to U(1)_Q$.

Notice that only *left-handed* quarks and leptons interact with $SU(2)_L$ gauge fields. The SM lagrangian has the symbolic form[8]

$$\mathcal{L}_{SM} \;=\; \mathcal{L}_{\rm kin} \;-\; V(\Phi) \;+\; \mathcal{L}_{\rm Yuk.} \;, \tag{218}$$

where

$$\begin{aligned} \mathcal{L}_{\rm kin} &= -\frac{1}{4}F_{mn}^2 - \frac{1}{4}(F_{mn}^a)^2 + |D_m\Phi|^2 \\ &+\bar{\Psi}_L i\gamma^m D_m\Psi_L + \bar{\Psi}_R i\gamma^m D_m\Psi_R \;, \end{aligned} \tag{219}$$

and

$$\begin{aligned} D_m\Psi_L &= (\partial_m - ig\frac{\tau_a}{2}A_m^a - ig'\frac{Y_L}{2}B_m)\Psi_L \\ D_m\Psi_R &= (\partial_m - ig'\frac{Y_R}{2}B_m)\Psi_R \;, \end{aligned} \tag{220}$$

whereas the Higgs potential has the form

$$V(\Phi) = -\mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 \;. \tag{221}$$

[8]No QCD gluons in what follows. For strong interactions, see the *QCD lectures* by Stephane Munier.

The Yukawa sector $\mathcal{L}_{\text{Yuk.}}$ will be discussed later on. With our conventions the electric charge is related to the third component of the isospin T_3 and to the hypercharge Y via

$$Q \;=\; T_3 + \frac{Y}{2}\;. \tag{222}$$

6.2 Weak mixing angles and gauge boson masses

With the help of an $SO(4)$ rotation, the Higgs vev can be written as

$$\Phi \;=\; \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}\;, \text{where}\;\; v^2 = \frac{\mu^2}{\lambda} \;\simeq (246\;\text{GeV})^2 \quad (\text{from experimental data})\;. \tag{223}$$

Gauge boson *masses* arise from the covariant derivative (**homework**)

$$\begin{aligned} |D_m\Phi|^2 &\to \frac{g^2v^2}{8}|A_m^{(1)} - iA_m^{(2)}|^2 + \frac{v^2}{8}|gA_m^{(3)} - g'B_m|^2 \\ &= \frac{g^2v^2}{4}W^{+,m}W_m^- \;+\; \frac{(g^2+g'^2)v^2}{8}Z^mZ_m\;, \end{aligned}$$

where the definitions and the masses of gauge bosons are

$$\begin{aligned} W_m^{\pm} &= \frac{1}{\sqrt{2}}(A_m^{(1)} \mp iA_m^{(2)}) \quad , \quad M_w = \frac{gv}{2}\;, \\ Z_m &= \frac{gA_m^{(3)} - g'B_m}{\sqrt{g^2+g'^2}} \quad , \quad M_z = \frac{v}{2}\sqrt{g^2+g'^2}\;, \\ A_m &= \frac{g'A_m^{(3)} + gB_m}{\sqrt{g^2+g'^2}} \quad , \quad M_A = 0\;. \end{aligned} \tag{224}$$

We now introduce the *electroweak angle*

$$\cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}} = \frac{M_w}{M_z}\;,\; \tan\theta_w = \frac{g'}{g}\;, \tag{225}$$

that rotates from the weak basis to the mass basis

$$\begin{pmatrix} Z_m \\ A_m \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} A_m^{(3)} \\ B_m \end{pmatrix}\;. \tag{226}$$

Notice that the ratio

$$\rho \equiv \frac{M_w^2}{M_z^2\cos^2\theta_w} = 1 \text{ at tree} - \text{level in the SM}\;. \tag{227}$$

The ρ parameter has quantum corrections in the SM, which are dominated by the top quark. Any experimental deviation from the SM value is a possible hint of new physics. Conversely, any model of new physics has to be able to produce a ρ parameter close to one, which is one of the *precision tests* of the Standard Model. This is a killer for most proposals of Beyond the Standard Model physics. For example, technicolor-like theories have difficulties in this respect (although there is no formal proof that they cannot accomodate precision data). Finally, the definition of the electric charge is $e = g\sin\theta_w$.

The W and Z gauge bosons were discovered in 1983 by the UA1,UA2 collaboration at CERN. Their masses are $M_W \simeq 80.4$ GeV, $M_W \simeq 91.2$ GeV.

6.3 Neutral and charged currents

The neutral and charged currents are defined as the fermion bilinears coupling to the charged (W)and neutral (Z) gauge fields. They are worked out starting from the fermionic kinetic terms.
Homework: With the definitions above, show that

$$D_m = \partial_m - igA_m^a \frac{\tau_a}{2} - ig'\frac{Y}{2}B_m = \partial_m - ieQA_m$$
$$-\frac{ig}{2\sqrt{2}}(W_m^+\tau_+ + W_m^-\tau_-) - \frac{ig}{\cos\theta_w}Z_m(T_3 - \sin^2\theta_w Q) \tag{228}$$

The fermionic currents are defined as

$$\mathcal{L} = \bar{\Psi}_i i\gamma^m\partial_m\Psi_i + \frac{g}{\sqrt{2}}(W_m^+J_W^{m,+} + W_m^-J_W^{m,-}) + \frac{g}{\cos\theta_w}Z_mJ_Z^m + eA_mJ_{em}^m \tag{229}$$

Homework : Using the quantum numbers of the quarks/leptons, show that

$$\begin{aligned}
J_W^{m,+} &= \bar{\nu}_L^i\gamma^m e_L^i + \bar{u}_L^i\gamma^m d_L^i \ (\text{charged current}) \\
J_W^{m,-} &= \bar{e}_L^i\gamma^m\nu_L^i + \bar{d}_L^i\gamma^m u_L^i\ , \\
J_{em}^m &= -\bar{e}^i\gamma^m e^i + \frac{2}{3}\bar{u}^i\gamma^m u^i - \frac{1}{3}\bar{d}^i\gamma^m d^i\ , \\
J_Z^m &= J_m^3 - sin^2\theta_w J_{em}^m = \frac{1}{2}\bar{\nu}_L^i\gamma^m\nu_L^i + (-\frac{1}{2} + \sin^2\theta_w)\bar{e}_L^i\gamma^m e_L^i + \sin^2\theta_w\bar{e}_R^i\gamma^m e_R^i \\
&+(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w)\bar{u}_L^i\gamma^m u_L^i - \frac{2}{3}\sin^2\theta_w\bar{u}_R^i\gamma^m u_R^i \\
&(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_w)\bar{d}_L^i\gamma^m d_L^i + \frac{1}{3}\sin^2\theta_w\bar{d}_R^i\gamma^m d_R^i \\
&= \frac{1}{2}\sum_i \bar{\Psi}_i\gamma^m(g_V^i - g_A^i\gamma_5)\Psi_i \quad , \quad (\text{neutral current})
\end{aligned} \tag{230}$$

where Ψ_i denote collectively all fermions (quarks and leptons, in Dirac notation) of the Standard Model and

$$g_V^i = I_3^i - 2Q_i\sin^2\theta_W \quad , \quad g_A^i = I_3^i \tag{231}$$

are the vector and axial fermionic couplings to the Z boson. At low energies $E << M_W, M_Z$, the exchange of W and Z bosons lead to the Fermi charged current four-fermion interaction, plus a similar neutral current interaction

$$\mathcal{L}_F \ = \ -2\sqrt{2}G_F\left[J_W^{m,+}J_{m,-}^W \ + \ \rho J_Z^m J_{m,Z}\right]\ , \tag{232}$$

where we defined the parameter $\rho = \frac{M_W^2}{M_Z^2\cos^2\theta_w}$ which, as we noticed in the previous paragraph, equals one at tree-level in the Standard Model and plays an important role in quantum corrections and constraints on new physics.

The weak interaction experiments allowed the experimental measure of the electroweak angle. Measure at energies cloze to the Z mass, it equals $\sin^2\theta_w \simeq 0.23$.

6.4 Fermion masses and the CKM matrix

Dirac mass terms in the SM are *not gauge invariant*, due to the chiral nature of elctroweak interactions. We can however write Yukawa-type interactions by using the Higgs field

$$-\mathcal{L}_{\text{Yuk}} \ = \ h_{ij}^u\bar{q}_L^i u_R^j\tilde{\Phi} + h_{ij}^d\bar{q}_L^i d_R^j\Phi \ + \ h_{ij}^e\bar{l}_L^i e_R^j\Phi\ , \tag{233}$$

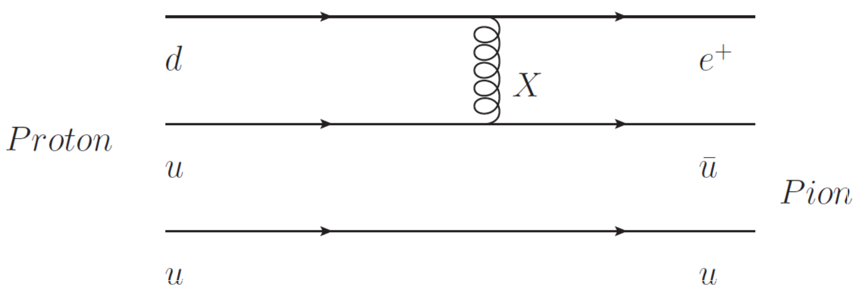


Fig. 11: Diagram leading to proton decay $p \to \pi^0 e^+$ in unified theories. The superheavy X particle is a GUT gauge boson.

where : $\tilde{\Phi} = \begin{pmatrix} \overline{\Phi}^0 \\ -\overline{\Phi}^+ \end{pmatrix}$ is the charge-conjugate Higgs field and $i,j = 1,2,3$ are flavor indices. The Yukawa couplings generate quarks and lepton masses after the electroweak symmetry breaking :

$$-\mathcal{L}_{\text{mass}} = m^u_{ij} \bar{u}^i_L u^j_R + m^d_{ij} \bar{d}^i_L d^j_R + m^l_{ij} \bar{e}^i_L e^j_R + \text{c.c.} \;, \tag{234}$$

where $m^{u,d,l}_{ij} = h^{u,d,l}_{ij} v/\sqrt{2}$. We use in what follows for compactness a matrix notation

$$-\mathcal{L}_{\text{mass}} = \bar{u}_L m^u u_R + \bar{d}_L m^d d_R + \bar{e}_L m^l e_R + \text{c.c.} \;, \tag{235}$$

where $m^{u,d,l}$ are 3×3 mass matrices in the flavor space.
Observation : The SM lagrangian has some automatic (consequences of the gauge symmetries) *global symmetries* :

$$\begin{aligned} &\text{baryon number} \quad U(1)_B \quad , \\ &\text{lepton numbers}\; U(1)_e \;,\; U(1)_\mu \;,\; U(1)_\tau \;. \end{aligned} \tag{236}$$

This is actually very *fortunate* since there are *very strong experimental constraints* on baryon and lepton number violating processes, for example :

$$\begin{aligned} &\text{proton lifetime} \quad \tau_p \;\geq\; 10^{33} \;\text{years} \;, \\ &\text{BR}(\mu \to e\gamma) < 2.4 \times 10^{-12} \;,\; \text{BR}(\mu^- \to e^- e^- e^+) < 10^{-10} \;. \\ &\text{BR}(B \to X_s \gamma) \;\sim 10^{-4} \;\Rightarrow\; b \to s\gamma \;\text{should be suppressed} \;. \end{aligned} \tag{237}$$

These limits constrain seriously any higher-dimensional operator violating flavor, generated by eventual new physics. For example, consider the operator

$$\mathcal{L}_{\text{eff}} \;\sim\; \frac{1}{M_X^2} (\bar{q}\gamma_m u^c_R)\; (\bar{l}\gamma^m d^c_R) \;. \tag{238}$$

Proton stability constrains the mass to be heavier than about $\Rightarrow M_X \geq 3 \times 10^{16}$ GeV. There is a long list of similar effective operators that are tightly constrained by the data. Another simple example is :

$$\mathcal{L}_{\text{eff}} \;\sim\; \frac{1}{M^2} (\bar{l}_2 \gamma_m l_1)\; (\bar{l}_1 \gamma^m l_1) \;\to\; \frac{1}{M^2} (\bar{\mu}\gamma_m e)\; (\bar{e}\gamma^m e) \;, \tag{239}$$

that can be generated by a *flavor-dependent Z' gauge boson*. The mass scale M is constrained by the limits on $\mu^- \to e^- e^- e^+$ to be heavier than $M > 1000$ TeV.
It turns out that almost all SM extension generates dangerous FCNC and/or proton decay, unless special

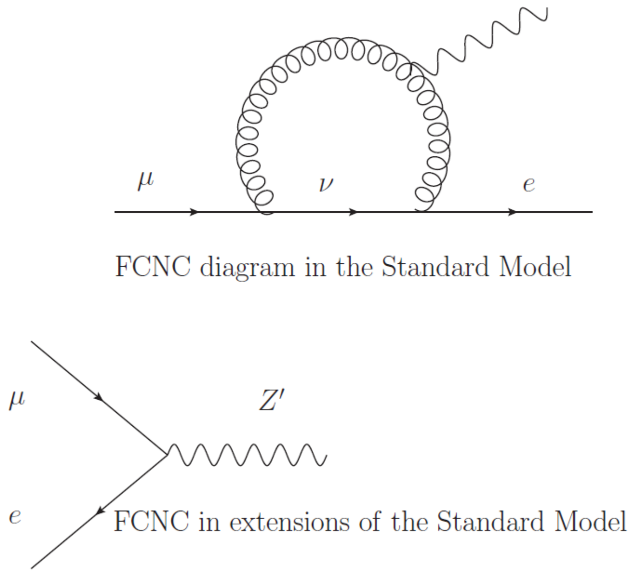


Fig. 12: Typical diagram in theories with a new neutral gauge boson, generating FCNC at tree-level.

structure.
For example, in *MSSM* we have to impose
- *R-parity*
- flavor blindness of soft terms.

Observation: With the field content of the SM, there is no operator generating neutrino masses at the renormalizable level. The main effective operator in the SM leading to neutrino masses is dimension five

$$\frac{h_{ij}^{\nu}}{M} \, (\bar{l}_i^c \Phi) \, (l_j \Phi) \quad \Rightarrow \quad m_{ij}^{\nu} \; = \; h_{ij}^{\nu} \, \frac{v^2}{M} \; . \tag{240}$$

Tiny values (of order 10^{-2} eV) neutrino masses ask then for $10^{12} \; GeV \; < M \; < 10^{15} \; GeV$. We will come back in more details to the problem of neutrino masses in Section (11).

Coming back to the quarks and charged leptons masses, we can define the *mass eigenstate basis* (as compared to the *weak eigenstate basis*) with the help of the 3×3 unitary transformations[9]

$$u_{L,R} \; = \; V_{L,R}^u u'_{L,R} \; , \; d_{L,R} \; = \; V_{L,R}^d d'_{L,R} \; , \; e_{L,R} \; = \; V_{L,R}^e e'_{L,R} \; , \tag{241}$$

such that

$$(V_L^u)^{\dagger} m^u V_R^u \; = \; diag \; (m_u, m_c, m_t) \; , \; \text{etc} \; . \tag{242}$$

In the mass basis, the neutral and the e.m. currents remain the same. To an excellent approximation, the neutrinos are massless in which case one is free to perform the same unitary transformation on the neutrinos as their $SU(2)_L$ charged lepton partners $\nu_L \; = \; V_L^e \nu'_L$. In the new basis, all SM lagrangian preserves the leptonic number per species L_e, L_μ, L_τ. These conservation laws are indeed observed experimentally with a great accuracy: no transition of the type $\mu \to e\gamma$, for example was observed until now. On the other hand, the story for the quark sector is different. The hadronic charged current becomes

$$(J_W^{m,+})_{\text{quarks}} \; \to \; \frac{1}{\sqrt{2}} \bar{u}'_L \gamma^m V_{CKM} d'_L \equiv \frac{1}{\sqrt{2}} \bar{u}'_L \gamma^m \tilde{d}_L \; , \tag{243}$$

[9] These transformations are not innocent; there is a quantum anomaly that we will discuss later on.

where $V_{CKM} = (V_L^u)^\dagger V_L^d$ is the (unitary) *CKM matrix* [29]. We also defined

$$\tilde{d}_L = V_{CKM} d'_L \ \leftrightarrow \ \begin{pmatrix} \tilde{d}_L \\ \tilde{s}_L \\ \tilde{b}_L \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} \tag{244}$$

There are therefore *flavor changing transitions* in the SM : $s \to uW^-$, etc. Experimental measurements give a *hierarchical form* of V_{CKM} of the type (Wolfenstein parametrization)

$$\begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} , \tag{245}$$

where $\lambda = \sin\theta_c \simeq .0.22$ is the Cabibbo angle. N. Cabibbo wrote first in 1962 the 2×2 version of the CKM matrix

$$\begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} . \tag{246}$$

It is simply to check that, after field redefinitions, V_{CKM} contain *three rotation angles* and a *CP violating phase*[10]. Notice also that CP violation in the SM is *suppressed* by λ^3 in V_{CKM}.
The *unitarity* of the CKM matrix

$$V_{ik} V^*_{jk} = \delta_{ij} \quad , \quad V^*_{ki} V_{kj} = \delta_{ij} \tag{247}$$

has various important consequences. One of them is *the GIM mechanism* (Glashow-Iliopoulos-Maiani, 1972), to which we will turn soon.

6.5 Higgs couplings

The Higgs mechanism described above is the minimal option, by using only one Higgs doublet Φ. Starting from the Standard Model lagrangian (218), one can easily work out the Higgs boson couplings to fermions and gauge fields in this gauge, by starting from the unitary gauge, where

$$\Phi = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} . \tag{248}$$

By replacing this into (218), one finds the Higgs couplings to gauge fields, fermions and itself are

$$\begin{aligned} \mathcal{L}_{\text{higgs couplings}} &= m_W^2 (1+\frac{h}{v})^2 W^+_\mu W^{\mu,-} + m_Z^2 (1+\frac{h}{v})^2 Z_\mu Z^\mu - \frac{1}{2} m_h^2 (1+\frac{h}{2v})^2 h^2 \\ &- \left[(1+\frac{h}{v}) \bar{u}_L m^u u_R + (1+\frac{h}{v}) \bar{d}_L m^d d_R + (1+\frac{h}{v}) \bar{l} m^l e_R + \text{h.c.} \right] . \end{aligned} \tag{249}$$

Obviously, the test of the SM nature of the Higgs boson sector is the proportionality between the SM Higgs boson couplings to the mass of the particles it interacts to. This linear proportionality relation ceases to be valid if the Higgs sector is non-minimal, containing more doublets or other representations.

Notice also that the diagonalization of the fermion mass matrices in the SM Higgs case diagonalize simultaneously the Higgs couplings to fermions. There are therefore no flavor transitions mediated for the minimal case of the SM Higgs doublet, which is very welcome in light of the tight constraints from

[10] In the case of N generations, a simple counting predicts $N(N-1)/2$ rotation angles and $(N-1)(N-2)/2$ CP violating phases. In fourth-generation extensions of the Standard Model, we therefore expect new sources of CP violations and apparent violations of the unitarity of the usual CKM matrix.

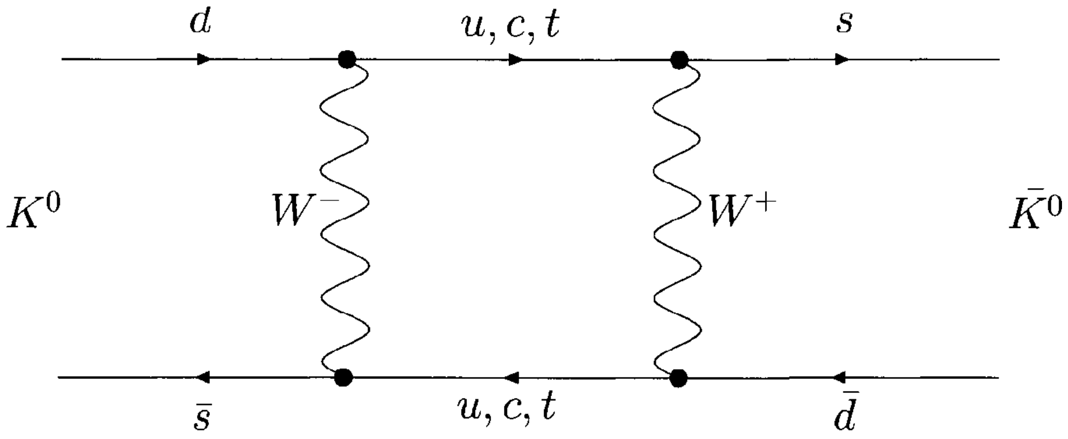


Fig. 13: K^0-$\bar{K}^0$ mixing generated at loop level in the Standard Model, with quarks $u_i = u, c, t$ running in the loop.

FCNC processes. It is straightforward to verify that if two Higgs doublets Φ_1, Φ_2 couple simultaneously to the same type type of quarks or leptons, like for example

$$-\mathcal{L}'_{\text{Yuk}} = \bar{q}_L^i u_R^j (h_{ij}^{1,u} \Phi_1 + h_{ij}^{2,u} \Phi_2) , \tag{250}$$

then generically it is not possible anymore to diagonalize simultaneously the fermion mass matrices and the fermion couplings to the Higgs scalars. The simplest models with no Higgs-induced FCNC effects in multi-Higgs extensions of the Standard Model are those for which the three generations of the same type of quarks (or leptons) couple to just one Higgs doublet. For example for two-Higgs doublet models H_1, H_2, such a model contains Yukawas of the type

$$-\mathcal{L}_{\text{Yuk}} = h_{ij}^u \bar{q}_L^i u_R^j H_2 + h_{ij}^d \bar{q}_L^i d_R^j H_1 \ + \ h_{ij}^e \bar{l}_L^i e_R^j H_1 \ . \tag{251}$$

6.6 The GIM mechanism

The FCNC (flavor changing neutral currents) effects were measured experimentally to be small. This was puzzling in the 1970's, but it was explained in the SM by GIM [23]. Consider for ex. the $K^0 - \bar{K}^0$ mixing, which can arise at the loop-level.

The amplitude of the process has the form

$$A_{K^0\bar{K}^0} \sim \frac{g^4}{M_W^2} (\sum_i V_{id} V_{is}^*)(\sum_j V_{js}^* V_{jd}) F(x_i, x_j) \ , \tag{252}$$

where $x_i = \frac{m_i^2}{M_W^2}$ and F is a (loop) function depending on the ratio of up-type quark masses and the W mass. Let us define in what follows $\xi_i = V_{si}^\dagger V_{id}$. Unitarity of the CKM matrix implies $\sum_i \xi_i = 0$. The computation can be performed in an arbitrary R_ξ gauge, taking into account the contribution of the massive gauge bosons and Goldstone's in the loop. It turns out however, that due to the good UV convergence properties and the unitarity of the CKM matrix, it is possible to perform the computation in the unitary gauge $\xi \to \infty$. To the lowest order in an expansion in the external momenta, the amplitude of the process in the unitary gauge is

$$A_{K^0\bar{K}^0} = \frac{ig^4}{2} \sum_{i,j} \xi_i \xi_j \int \frac{d^4k}{(2\pi)^4} \left[\bar{u}_{s,L} \gamma^\alpha \frac{1}{\not{k} - m_i} \gamma^\nu u_{d,L} \right] \frac{\eta_{\alpha\beta} - \frac{k_\alpha k_\beta}{M_W^2}}{k^2 - M_W^2} \frac{\eta_{\mu\nu} - \frac{k_\mu k_\nu}{M_W^2}}{k^2 - M_W^2} \left[\bar{v}_{s,L} \gamma^\mu \frac{1}{\not{k} - m_j} \gamma^\beta v_{d,L} \right] . \tag{253}$$

If one neglects the up-quark mass, one can replace

$$\sum_{i=u,c,t} \xi_i \frac{1}{\not{k} - m_i} \;=\; \sum_{i=c,t} \xi_i \left(\frac{1}{\not{k} - m_i} - \frac{1}{\not{k}}\right) . \tag{254}$$

After working out the Dirac matrix algebra, one can rewrite the amplitude in the simpler way

$$A_{K^0 \bar{K}^0} = \frac{g^4}{2M_W^2} \sum_{i,j=c,t} \xi_i \xi_j F_{ij} \left(\bar{u}_{s,L} \gamma^\mu u_{d,L}\right) \left(\bar{v}_{s,L} \gamma_\mu v_{d,L}\right) . \tag{255}$$

where

$$F_{ij} = i M_W^2 \int \frac{d^4k}{(2\pi)^4} \frac{1 - 2\frac{k^2}{M_W^2} + \frac{(k^2)^2}{4M_W^4}}{k^2 (k^2 - M_W^2)^2} \frac{m_i^2}{k^2 - m_i^2} \frac{m_j^2}{k^2 - m_j^2} . \tag{256}$$

An explicit calculation yields for $i \neq j$ the symmetric dimensionless function

$$F(x_i, x_j) = \frac{x_i x_j}{x_i - x_j} \left[(1 - 2x_i + \frac{1}{4} x_i^2) \frac{\ln x_i}{(1 - x_i)^2} + (i \leftrightarrow j) \right] - \frac{3 x_i x_j}{4(1 - x_i)(1 - x_j)} \tag{257}$$

and

$$F(x_i, x_i) = \frac{3}{2} \left(\frac{x_i}{1 - x_i} \right)^3 \ln x_i - \frac{x_i (4 - 11 x_i + x_i^2)}{4(1 - x_i)^2} . \tag{258}$$

In the limit of equal or vanishing quark masses, the amplitude vanishes due to the unitarity of V_{CKM} :

$$A_{K^0 \bar{K}^0} \sim \frac{g^4}{M_W^2} (\sum_i V_{id} V_{is}^*)(\sum_j V_{js}^* V_{jd}) F(x, x) = 0 . \tag{259}$$

Let us define

$$\mathcal{F}_K = (\sum_i V_{id} V_{is}^*)(\sum_j V_{js}^* V_{jd}) F(x_i, x_j) . \tag{260}$$

Applying the Wolfenstein parametrization for the CKM matrix and neglecting for simplicity the up-quark mass, one can write

$$\mathcal{F}_K = \lambda^2 (1 - \frac{\lambda^2}{2})^2 F(x_c, x_c) + 2A^2 \lambda^6 (1 - \rho + i\eta) F(x_c, x_t) + A^2 \lambda^{10} (1 - \rho + i\eta)^2 F(x_t, x_t) . \tag{261}$$

Numerically the main contribution is proportional to $g^4 \lambda^2 (m_c^2 - m_u^2)/M_W^4$ and is in good agreement with the experimental result. The CP violation in the amplitude comes from a virtual propagator of the top quark. The Feynman diagram for the $K^0 - \bar{K}^0$ mixing can also be expressed as a dimension-six operator

$$\mathcal{L}_{\text{eff.}}^{\Delta S=2} = \frac{G_F}{\sqrt{2}} g_2^2 (\bar{d}_L \gamma_m s_L)(\bar{d}_L \gamma^m s_L) \mathcal{F}_K . \tag{262}$$

The mass difference between the physical mass eigenstates is then equal to

$$\delta m_K = -2 \,\mathrm{Re}\, \langle K^0 | \mathcal{L}_{\text{eff.}}^{\Delta S=2} | \bar{K}^0 \rangle = \frac{G_F}{\sqrt{2}} \, g_2^2 \,\mathrm{Re}\, \langle K^0 | (\bar{d}_L \gamma_m s_L)(\bar{d}_L \gamma^m s_L) | \bar{K}^0 \rangle \mathcal{F}_K . \tag{263}$$

The original computation [32] used the vacuum saturation approximation

$$\langle K^0 | (\bar{d}_L \gamma_m s_L)(\bar{d}_L \gamma^m s_L) | \bar{K}^0 \rangle \simeq \langle K^0 | (\bar{d}_L \gamma_m s_L) | 0 \rangle \langle 0 | (\bar{d}_L \gamma^m s_L) | \bar{K}^0 \rangle \tag{264}$$

and the pseudo-Goldstone relation

$$\langle K^0(p) | (\bar{d}_L \gamma_m s_L) | 0 \rangle = p_m \frac{f_K}{\sqrt{2 m_K}} , \tag{265}$$

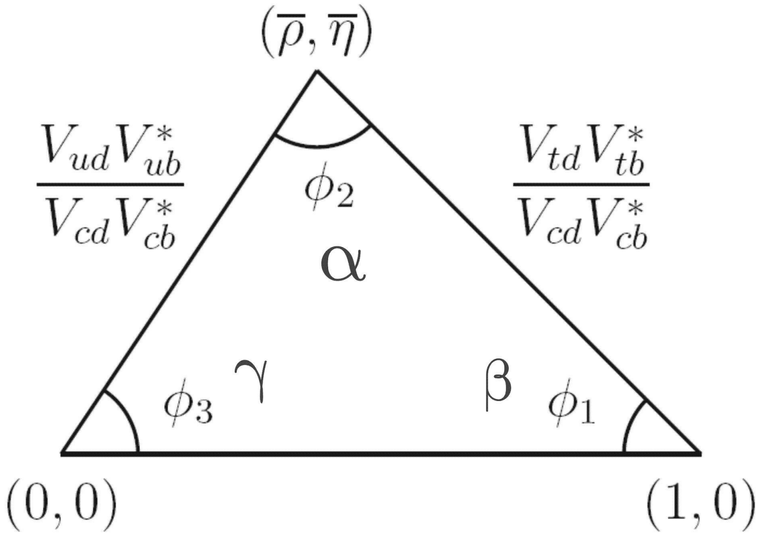


Fig. 14: The unitarity triangle. From [31].

similar to (159), f_K is the Kaon decay constant. Due to strong interaction effects, today one simply adds a fudge factor to the amplitude called B_K. One therefore finds

$$\delta m_K = \frac{G_F}{\sqrt{2}} B_K f_K^2 m_K \mathcal{F}_K \ , \tag{266}$$

A similar analysis can be applied to the $B^0 - \bar{B}^0$ mixing. The result is

$$\mathcal{F}_B = A^2\lambda^6 F(x_c, x_c) + 2A^2\lambda^6(1-\rho+i\eta)F(x_c, x_t) + A^2\lambda^6(1-\rho+i\eta)^2 F(x_t, x_t) \ . \tag{267}$$

In this case, it is the top quark contribution which dominates the mass difference.

Historical Remark : In 1972, only the u, d and s quarks were known. The GIM mechanism is considered to be the first convincing proof of the existence of the charm quark.

Homework : Write down explicitly the diagrams for the $K^0 - \bar{K}^0$ mixing in the two generation case, with u and c quarks in the loop.

The unitarity relation

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* \ = \ 0 \tag{268}$$

can be represented geometrically as a *triangle* in a plane $\Rightarrow$ *unitarity triangle*. It is customary to rescale the length of one side, i.e. $|V_{cd}V_{cb}^*|$ (well-known), to 1 and to align it along the real axis. The angles are defined as

$$\beta = \arg(-\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}) \ , \ \gamma = \arg(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}) \tag{269}$$

and the lengths are

$$R_t \ = \ |\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}| \quad , \quad R_u \ = \ |\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}| \ . \tag{270}$$

On the other hand, quarks, leptons masses and the CKM matrix feature strong hierarchies. For example, from neutrino masses to the top mass there are 10^{11} orders of magnitude $m_\nu \sim 10^{-2}\ eV\ << m_e = 0.511\ MeV << m_t \sim 172\ GeV$!

There is *no hint for a solution of this flavor puzzle* in the SM, since the Yukawa couplings are free-parameters and therefore are not predicted. We are clearly *missing something* : maybe an additional global or gauge symmetry [30] or maybe this comes from an extra dimensional localization or environmental selection.

6.7 The custodial symmetry

The tree-level relation $\rho = M_W^2/(M_Z^2 \cos^2\theta_w) = 1$ can be understood as the result of an *(approximate) symmetry*, called *custodial symmetry*, proven by Sikivie,Susskind,Voloshin and Zakharov [33] .

Theorem *: In any theory of electroweak interactions which conserves the electric charge and has an approximate global $SU(2)$ symmetry under which A^a_m transform as a triplet, $\rho = 1$ at tree-level.*

Here approximate means in the limit of zero hypercharge coupling $g' = 0$ and in the absence of the Yukawa couplings.

Proof: Under the assumptions above, the gauge boson mass matrix is of the form

$$\begin{pmatrix} M^2 & 0 & 0 & 0 \\ 0 & M^2 & 0 & 0 \\ 0 & 0 & M^2 & m_1^2 \\ 0 & 0 & m_1^2 & m_2^2 \end{pmatrix} . \tag{271}$$

The masslessness of the photon implies $M^2 m_2^2 - m_1^4 = 0$. The resulting $W_3 - A$ mass matrix, written in terms of the W and Z masses, is then of the form :

$$\begin{pmatrix} M_W^2 & \pm M_W\sqrt{M_Z^2 - M_W^2} \\ \pm M_W\sqrt{M_Z^2 - M_W^2} & M_Z^2 - M_W^2 \end{pmatrix} . \tag{272}$$

It is then easy to check that $M_W = \cos\theta_w M_Z$.

On the other hand, in the SM the Higgs potential $V(\Phi^\dagger\Phi)$ is invariant under an $SO(4)$ global symmetry. Indeed, let us write explicitly the four real components of the SM Higgs doublet

$$\Phi = \begin{pmatrix} \Phi_1 + i\Phi_2 \\ \Phi_3 + i\Phi_4 \end{pmatrix} \quad , \text{ then } \quad \Phi^\dagger\Phi = \sum_{i=1}^4 \Phi_i^2 \qquad . \tag{273}$$

It is then transparent that the Higgs potential and kinetic term have an $SO(4) = SU(2)_L \times SU(2)_R$ symmetry. The Higgs vev

$$\Phi = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \text{ breaks } SO(4) \to SO(3) \ = \ SU(2)_D \ , \tag{274}$$

which corresponds precisely to the custodial symmetry. From these considerations, it is clear that not any Higgs representations preserve the custodial symmetry.

What happens for other Higgs representations ? It can be shown that, considering Higgs representations in weak isospin representations of isospin I, the rho parameter is given by

$$\rho = \frac{1}{2}\frac{\sum_i (I_i(I_i+1) - I_{3i}^2)\,|\langle 0|\phi_i|0\rangle|^2}{\sum_i I_{3i}^2\,|\langle 0|\phi_i|0\rangle|^2} \quad . \tag{275}$$

It is then easy to check that for an arbitrary number of singlet and higgs doublets, $\rho = 1$. On the other hand, for Higgs triplets for example, the higgs vev generate the breaking $SO(3) \to SO(2)$. In this case there is no custodial symmetry and $\rho \neq 1$.

Strong interactions preserve electric charge and strong isospin. A natural choice for the custodial symmetry is therefore the strong isospin, which then guarantees that $\rho = 1$ to all order in the strong interactions.

A *useful parametrization* for estimating the violation of the custodial symmetry is :

$$\mathcal{H} = \begin{pmatrix} i\tau_2\Phi^* & \Phi \end{pmatrix} = \begin{pmatrix} \Phi_0^* & \Phi_+ \\ -\Phi_+^* & \Phi_0 \end{pmatrix} \ , \ \Phi^\dagger\Phi \ = \ \frac{1}{2}\, Tr\mathcal{H}^\dagger\mathcal{H} \, . \tag{276}$$

$V(\Phi^\dagger\Phi)$ is invariant under $\mathcal{H} \ \to \ U_L\mathcal{H}U_R^\dagger$, with $U_{L,R}$ being 2×2 unitary matrices implementing $SU(2)_L \times SU(2)_R$ transformations. The electroweak symmetry breaking pattern is then

$$\langle\mathcal{H}\rangle = \frac{v}{\sqrt{2}} I_{2\times 2} \quad \text{breaks} \quad SU(2)_L \times SU(2)_R \to SU(2)_D \, . \tag{277}$$

As anticipated, the hypercharge gauge interactions $U(1)_Y$ and Yukawa couplings *break* the custodial symmetry. However the particular coupling

$$\mathcal{L}_{\text{Yuk}} \ = \ h \ \begin{pmatrix} \bar{t}_L & \bar{b}_L \end{pmatrix} \mathcal{H} \begin{pmatrix} t_R \\ b_R \end{pmatrix} \tag{278}$$

is invariant under $SU(2)_D$. This corresponds to the limit of equal masses in the quark doublet $h_t = h_b$. On the other hand, W and Z boson masses have quantum corrections that lead to calculable deviations from $\rho = 1$. This can be understood as quantum corrections that change the mass matrix (271) to

$$\begin{pmatrix} \tilde{M}^2 & 0 & 0 & 0 \\ 0 & \tilde{M}^2 & 0 & 0 \\ 0 & 0 & \tilde{M}_3^2 & \tilde{m}_1^2 \\ 0 & 0 & \tilde{m}_1^2 & \tilde{m}_2^2 \end{pmatrix} , \tag{279}$$

where $\tilde{M}^2 = M^2 + \delta M^2$, etc, δM^2 being the quantum correction to the corresponding mass. The breaking of the custodial $SU(2)$ symmetry in the gauge mass matrix by quantum corrections is an important test of the quantum structure of the Standard Model. A recomputation of the ρ parameter in this case gives

$$\rho = \frac{\tilde{M}^2}{\tilde{M}_3^2} \ = \ 1 \ - \ \frac{i}{M_W^2} \left[\Pi_{++} - \Pi_{33}\right](0) \ , \tag{280}$$

where the mass difference between the charged and the neutral gauge fields component is computed from the vacuum polarization at zero momentum

$$(\Pi_{++}^{\mu\nu} - \Pi_{33}^{\mu\nu})(0) \equiv \eta^{\mu\nu}(\Pi_{++} - \Pi_{33})(0) = i\eta^{\mu\nu}(\tilde{M}^2 - \tilde{M}_3^2) \, . \tag{281}$$

Here we defined the vacuum polarization tensor starting from the inverse gauge boson propagator

$$D_{mn}^{-1}(q) = D_{0,mn}^{-1}(q) - \Pi_{mn}(q) \, . \tag{282}$$

Using the expression of the free gauge-field propagator

$$D_{0,mn}^{-1}(q) = i\left[\eta_{mn}(q^2 - M^2) - (1 - \frac{1}{\xi})q_m q_n\right] \ , \tag{283}$$

and defining in the case of $SU(2)$ gauge group $\Pi_{ij}^{mn}(0) = \eta^{mn}\Pi_{ij}(0)$, one finds

$$D_{mn}^{-1}(0) = -i\eta_{mn}\left[M^2 - i\Pi(0)\right] \ , \tag{284}$$

therefore one can indeed identify the gauge boson mass corrections as $\delta M_{ij}^2 = -i\Pi_{ij}(0)$. In the SM, the leading quantum contributions comes from the third generation of quarks. The one-loop expressions for the relevant vacuum polarization diagrams are

$$\Pi_{++}^{mn} = -3(\frac{ig}{\sqrt{2}})^2 \int \frac{d^4k}{(2\pi)^4} Tr \left[\gamma^m \frac{1-\gamma_5}{2} \frac{i}{\not{k} - m_t} \gamma^n \frac{1-\gamma_5}{2} \frac{i}{\not{k} - m_b}\right] ,$$
$$\Pi_{33}^{mn} = -3(\frac{ig}{2})^2 \int \frac{d^4k}{(2\pi)^4} Tr \left[\gamma^m \frac{1-\gamma_5}{2} \frac{i}{\not{k} - m_t} \gamma^n \frac{1-\gamma_5}{2} \frac{i}{\not{k} - m_t}\right] + (t \leftrightarrow b) . \quad (285)$$

An explicit evaluation of the relevant traces give

$$\Pi_{++}^{mn} = \frac{3g^2}{2} \eta^{mn} \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2 - m_t^2)(k^2 - m_b^2)} ,$$
$$\Pi_{33}^{mn} = \frac{3g^2}{4} \eta^{mn} \int \frac{d^4k}{(2\pi)^4} \left(\frac{k^2}{(k^2 - m_t^2)^2} - \frac{k^2}{(k^2 - m_b^2)^2} \right) . \quad (286)$$

After a Wick rotation and and explicit evaluation of the (UV finite) integral, one finds the final result [35]

$$\delta\rho \ = \ \frac{3g^2}{64\pi^2 M_W^2} \left[m_t^2 + m_b^2 - \frac{2m_t^2 m_b^2}{m_t^2 - m_b^2} \ln \frac{m_t^2}{m_b^2} \right] \ - \ \frac{3g'^2}{32\pi^2} \ln \frac{m_H}{M_Z} + \cdots \quad (287)$$

Notice that the first term on the rhs vanishes in the custodial limit $m_t = m_b$, as expected. The $\cdots$ in (287) are *subleading* contributions from the SM or from eventual new physics contributions that have to be smaller than 10^{-3} in order to fit the experimental data [34].

7 Quantum corrections and renormalization.

Quantum corrections through loops are subtle to incorporate, due to UV divergences appearing for large momenta of virtual particles running in the loops. Dealing with these divergences is crucial in order to extract physical results. This led to the program of renormalization, which was brilliantly confirmed by various precision measurements, in particular at the LEP collider. The proof of renormalization of the Standard Model led to the 1999 Nobel prize of G. 't Hooft and M. Veltman [36].

7.1 UV divergences and regularization.

Perturbation theory in Quantum Field Theory is plagued with *UV divergences*. We have to keep an UV cutoff Λ (which can be implemented in various different ways) in computing physical quantities. There are three cases that arise :
- *Super-renormalizable theories* : In this case only a finite number of Feynman diagrams diverge. Beyond a sufficiently large number of loops, all Green functions are finite.
- *Renormalizable theories* : a finite number of amplitudes/Green functions diverge, with a number of external legs below a maximal value (which is for example four for the ϕ^4 theory, three for QED and four for Yang-Mills theories). For these amplitudes, the UV divergences arise at all orders in perturbation theory.
- *Non-renormalizable theories* : All amplitudes, with an arbitrary number of legs are UV divergent at a certain order in perturbation theory.

In renormalizable and super-renormalizable theories, UV divergences can be *absorbed* into *rescaling of fields* and *redefinitions* of the various couplings and masses. Taking the couplings/masses from experimental data and "hiding" the UV cutoff in their redefinitions, we obtain physical quantities free of UV divergences. In this case, *the theory is predictive at any energy scale*.
In non-renormalizable theories, we need an *infinite number* of couplings and masses in order to absorbe

the UV divergences. We would need an infinite amount of experimental data to determine all these couplings. Therefore, at high-energies $E > \Lambda$ *the theory looses its predictive power*. However, at low-energy the theory is *perfectly predictive*. The typical example of this type is the General Relativity.

7.2 Relevant, marginal and irrelevant couplings

Consider a scalar theory of the form

$$S_\Lambda \;=\; \int d^4x \left(\frac{1}{2}(\partial\phi)^2 \;+\; \frac{m^2\phi^2}{2} \;+\; \sum_n \lambda_n \phi^n \right) \;, \tag{288}$$

where S_Λ is the euclidian action defined with a cutoff Λ. The couplings λ_n have (classical) mass dimensions $[\lambda_n] = 4 - n$. Let us consider the theory with two different maximal euclidian cutoff momenta :
i) $0 \;<\; p \;<\; \Lambda$
ii) $0 \;<\; p \;<\; \Lambda' \;=\; \epsilon\,\Lambda$, where $\epsilon < 1$.
In case ii) the theory has therefore a *lower cutoff* and it is interpreted as a theory where the high-momenta of theory i) were *integrated out*. The theory i) has the action (288). In the theory ii) the cutoff can be redefined to be the same as in i) with the help of a *scale transformation*

$$x' = \epsilon\, x \quad , \quad p' = \epsilon^{-1} p \quad , \quad \phi' = \epsilon^{-1}\phi \;. \tag{289}$$

In terms of the rescaled field and coordinates, the action of theory ii) become

$$S_{\Lambda'} \;=\; \int d^4x' \left(\frac{1}{2}(\partial'\phi')^2 \;+\; \frac{m'^2(\phi')^2}{2} \;+\; \sum_n \lambda'_n (\phi')^n \right) \;, \tag{290}$$

where

$$m'^2 \;=\; \frac{1}{\epsilon^2}\, m^2 \quad , \quad \lambda'_n \;=\; \epsilon^{n-4}\,\lambda_n \;. \tag{291}$$

Notice that the new mass and couplings scale with their classical dimension. We see therefore that the mass and couplings with positive dimension *grow* in the IR, whereas couplings with negative dimension *decrease* in the IR. It is said that

$$\begin{aligned} [\lambda_n] > 0 \quad &\Rightarrow \quad \text{relevant couplings} \;, \\ [\lambda_n] = 0 \quad &\Rightarrow \quad \text{marginal couplings} \;, \\ [\lambda_n] < 0 \quad &\Rightarrow \quad \text{irrelevant couplings} \;. \end{aligned} \tag{292}$$

This point of view on renormalization was introduced by K. Wilson and is summarized, for example, in [37].

7.3 (Non)renormalizability and couplings dimensions.

There is a straight connection between renormalizability and the three type of couplings previously defined :
- relevant couplings $\Rightarrow$ super-renormalizability.
- marginal couplings $\Rightarrow$ renormalizability.
- irrelevant couplings $\Rightarrow$ non-renormalizability.
It is easy to argue for this by *dimensional arguments*. Let us consider some simple examples, going back in Minkowski space :
a) - Relevant coupling

$$\mathcal{L} \;=\; \frac{1}{2}(\partial\phi)^2 \;-\; \frac{m^2\phi^2}{2} \;-\; \lambda_3\phi^3 \;. \tag{293}$$

The coupling has dimension $[\lambda_3] = +1$, so it is relevant. At one-loop, the UV divergent terms lead to new terms in the lagrangian (homework:)

$$\delta\mathcal{L}_1 \sim \lambda_3 \Lambda^2 \phi + \lambda_3^2 \phi^2 \ln\Lambda \ , \tag{294}$$

which are both of super-renormalizable type. The first leads to a scalar tadpole, whereas the second leads to a mass renormalization. At two loops, the only UV divergences are a cosmological constant and a scalar tadpole. At three loops, there is only a log UV divergence in the cosmological constant. No UV divergences exist at higher loops.
Dimensional argument : By dimensional analysis, the highest UV divergent term in the coupling is the three-loop vacuum energy

$$\lambda_3^4 \ \ln\Lambda \ . \tag{295}$$

Higher loops have higher powers in λ_3 and cannot contribute to the UV divergent terms in the effective lagrangian.
Observation: $1/m^2$ terms are IR, not UV contributions, so they cannot appear in UV divergent terms.
b) - Irrelevant coupling

$$\mathcal{L} \ = \ \frac{1}{2}(\partial\phi)^2 \ - \ \frac{m^2\phi^2}{2} \ - \ \lambda_6 \phi^6 \ . \tag{296}$$

The coupling has dimension $[\lambda_6] = -2$, so it is irrelevant. At one-loop, the UV divergent terms in the eight-point amplitude lead to (homework) :

$$\Gamma^{(8)}_{1-\text{loop}}(p_i) \ \sim c\, \lambda_6^2 \ \ln\Lambda + \cdots \ . \tag{297}$$

To cancel this divergence, one has to add a new coupling to the original action

$$\delta\mathcal{L}_1 \ \sim \ \lambda_8 \, \phi^8 \ , \tag{298}$$

and to *adjust the coupling* λ_8 such that

$$\lambda_8 \ + \ c\, \lambda_6^2 \ \ln\Lambda \ = \ \text{finite} \ . \tag{299}$$

At two-loops, we get new *new UV divergences*, like the one in the six-point amplitude, proportional to

$$\Gamma^{(6)}_{2-\text{loops}}(p_i) \ \sim \ c' \ (p_i p_j) \ \lambda_6^2 \ \ln\Lambda \ , \tag{300}$$

which can be canceled by adding *another coupling*

$$\delta\mathcal{L}_2 \ \sim \ \lambda_8' \, \phi^4 (\partial\phi)^2 \ , \tag{301}$$

such that

$$\lambda_8' \ + \ c' \, \lambda_6^2 \ \ln\Lambda \ = \ \text{finite} \ . \tag{302}$$

The UV divergences *proliferate* at higher loop orders, generating an infinite tower of operators of higher and higher dimension.
Dimensional argument: Terms of the type $\lambda_6^n \phi^{4+2n} \ln\Lambda$, $\lambda_6^n (\partial\phi)^2 \phi^{2n} \ln\Lambda$ have the correct dimension to be generated for any n. Predictivity at high-energy is *lost*. Let us however define $\lambda_6 \sim 1/M^2$. Then : In the IR $E < M$, the effect of non-renormalizable operators on physical quantities is proportional to some positive power or E/M and/or m/M, so their effects is negligible.
Effective theories with cutoff Λ (ex. General relativity, $\Lambda = M_P$) are therefore predictive at energies $E << \Lambda$.

Another viewpoint on this problem is the following: for $\mathcal{L}_{\text{int}} \ = \ \sum_n \lambda_n \phi^n$, the leading cross-section for $2 \to 2$ particle scattering is

$$\sigma \ = \ \sum_n c_n \lambda_n^2 E^{2n-10} \ \sim \frac{1}{E^2} \sum_n c_n (\frac{E}{M})^{2n} \ , \tag{303}$$

for $\lambda_n \sim 1/M^{n-4}$. Therefore the *predictive power is lost* for $E \geq M$.

7.4 Coupling constant renormalization for ϕ^4 theory.

Consider the ϕ^4 theory of lagrangian

$$\mathcal{L} \;=\; \frac{1}{2}(\partial\phi)^2 - \frac{m_0^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4\;. \tag{304}$$

Let us compute the four-point function at one-loop. By using the Feynman rules for the ϕ^4 theory, we find, according to the figure in the next page

$$\begin{aligned}\Gamma^{(4)}(k_1k_2k_3k_4) \;&=\; -i\lambda_0 \;+\; \frac{(-i\lambda_0)^2}{2}\times\\ \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m_0^2}\,\frac{i}{(p-k_1-k_2)^2-m_0^2} \;&+\; \text{two crossing terms}\;.\end{aligned} \tag{305}$$

After the Wick rotation to euclidian momenta, the result is given by

$$\begin{aligned}&\Gamma^{(4)}(k_1k_2k_3k_4) = -i\lambda_0 + \frac{i\lambda_0^2}{2}\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2+m_0^2}\,\frac{1}{(p-k_1-k_2)^2+m_0^2}\\ &+\text{two crossing terms}\;.\end{aligned} \tag{306}$$

The integral is log divergent in the UV. There are various ways to "renormalize" the integral. Here is a simple way. Define

$$V(s)\equiv\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2+m_0^2}\,\frac{1}{(p-k_1-k_2)^2+m_0^2}\;=\;\int_{p^2\geq\mu^2}^{\Lambda}\frac{d^4p}{(2\pi)^4}\,\frac{1}{p^4}+\text{finite}\;, \tag{307}$$

where the energy scale μ is arbitrary. We find (homework)

$$\begin{aligned}\Gamma^{(4)}(k_1k_2k_3k_4) &= -i\lambda_0 + \frac{i\lambda_0^2}{2}[V(s)+V(t)+V(u)]\\ &=\; -i\lambda_0 + \frac{3i\lambda_0^2}{16\pi^2}\,\ln\frac{\Lambda}{\mu}+\text{finite} \;=\; -i\lambda(\mu)+\text{finite}\;.\end{aligned} \tag{308}$$

What is the *physical interpretation* of this manipulation ? We can separate the answer into two separate steps:

i) λ_0 is *not a physical parameter*. It can be chosen to depend on Λ such that

$$\lambda(\mu) = \lambda_0(\Lambda) - \frac{3\lambda_0^2}{16\pi^2}\ln\frac{\Lambda}{\mu} \tag{309}$$

is independent of Λ.

ii) Any value of μ leads to the same physical result. We can find a differential equation for λ by using the fact that λ_0 is independent of μ. We obtain

$$\frac{d\lambda}{d\ln\mu} \;=\; \frac{3\lambda^2}{16\pi^2} \;=\; \beta(\lambda)\;, \tag{310}$$

which is called the *renormalization group equation (RGE)* of λ at one-loop, with $\beta(\lambda)=\frac{3\lambda^2}{16\pi^2}$ being the one-loop RG *beta function* coefficient. The solution of (325) is (homework)

$$\lambda(\mu) \;=\; \frac{\lambda(\mu_0)}{1-\frac{3\lambda(\mu_0)}{16\pi^2}\ln\frac{\mu}{\mu_0}}\;. \tag{311}$$

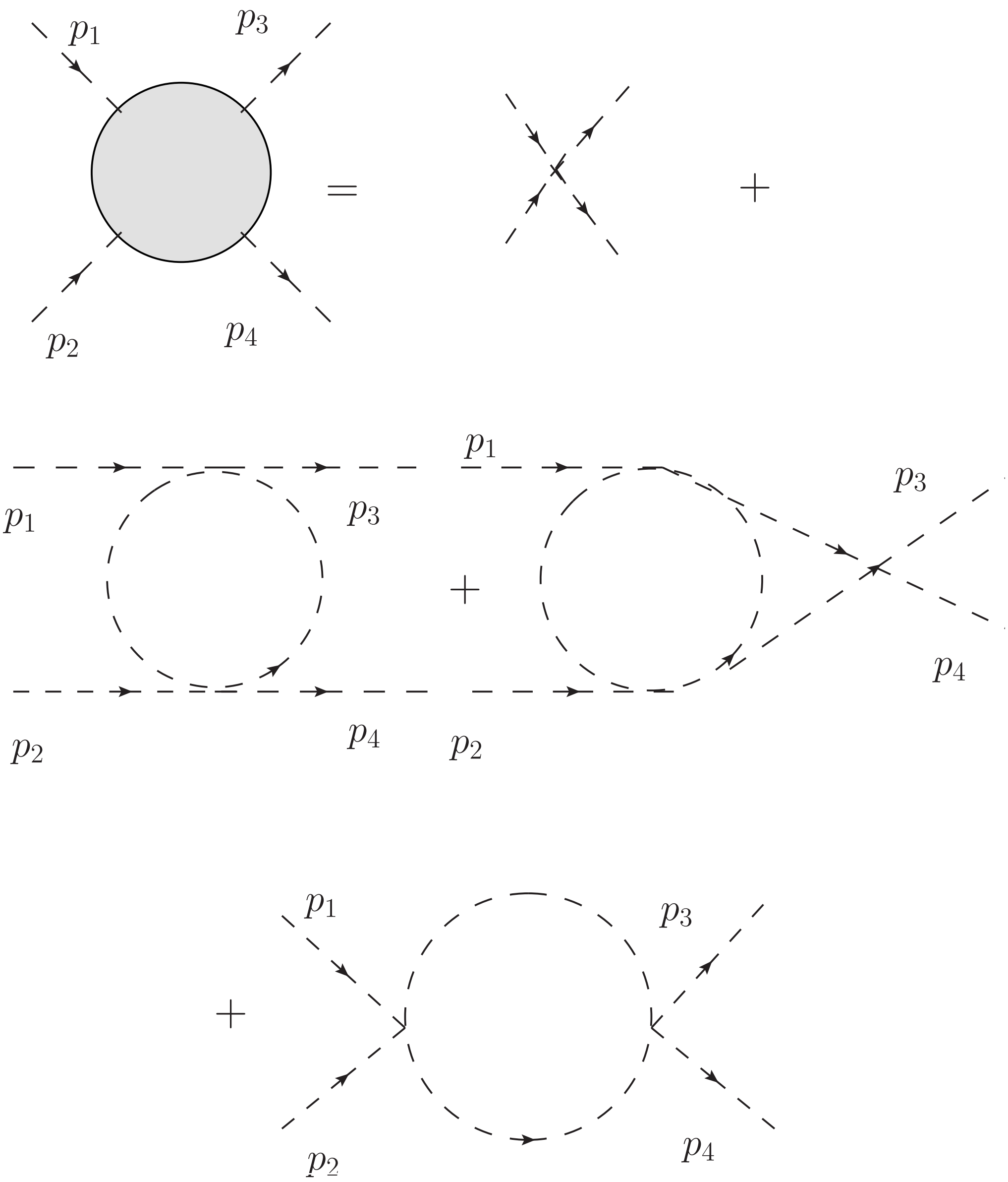


Fig. 15: Diagrams contributing to the renormalization of the self-coupling λ in ϕ^4 theory.

7.5 Bare parameters versus counterterms

Notice that there is an equivalent viewpoint to the renormalization procedure just described : to "add" a local *"counterterm"* to the lagrangian

$$\mathcal{L}_r \,+\, \delta\mathcal{L} \,=\, \mathcal{L}\,, \tag{312}$$

to cancels the UV divergence. In order to illustrate the procedure, let us consider again the ϕ^4 theory of lagrangian (304). When computing the propagator of the scalar field, the residue of the propagator at the pole Z, called wave function renormalization, defined as

$$\int d^4x\, e^{ipx}\, \langle 0|T\phi(x)\phi(0)|\rangle = \frac{iZ}{p^2 - m^2} + \text{terms regular at } p^2 = m^2 \tag{313}$$

is usually divergent. We can then define a "renormalized" field with unit residue, via $\phi = Z^{1/2}\phi^r$, such that the lagrangian (304) becomes

$$\mathcal{L} \,=\, \frac{1}{2}Z(\partial\phi_r)^2 - \frac{Zm_0^2}{2}\phi_r^2 - \frac{Z^2\lambda_0}{4!}\phi_r^4\,. \tag{314}$$

One can avoid talking about bare parameters by splitting the lagrangian into a "renormalized" lagrangian and "counterterms

$$\begin{aligned}\mathcal{L} \,=\, &\frac{1}{2}(\partial\phi_r)^2 - \frac{m^2}{2}\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \\ &+\frac{1}{2}\delta Z(\partial\phi_r)^2 - \frac{\delta m^2}{2}\phi_r^2 - \frac{\delta\lambda}{4!}\phi_r^4\,,\end{aligned} \tag{315}$$

where

$$\delta Z = Z - 1 \quad , \quad \delta m^2 = Zm_0^2 - m^2 \quad , \quad \delta\lambda = Z^2\lambda_0 - \lambda\,. \tag{316}$$

The counterterms in the second line are treated as interactions in perturbation theory and are in order to satisfy some "renormalization conditions". In the case at hand, they can be defined as

$$\begin{aligned}&\Gamma^{(2)}(p^2 = m^2) = 0 \quad , \quad \frac{\partial\Gamma^{(2)}}{\partial p^2}(p^2 = m^2) = 1\,, \\ &\Gamma^{(4)}(s = 4m^2, t = u = 0) = -i\lambda\,.\end{aligned} \tag{317}$$

Once renormalization conditions are imposed, counterterms are determined and any divergences are eliminated; physical results expressed in terms of physical parameters are UV finite. This viewpoint is called renormalized perturbation theory, whereas the previous approach with bare parameters could be called bare perturbation theory. In analogy with the bare parameters viewpoint, there is no unique definition of the renormalized coupling λ. Another example of definition could be

$$\Gamma^{(4)}(s = t = u = 4m^2/3) = -i\lambda'\,. \tag{318}$$

The ambiguity in defining λ is precisely the one leading to the RGE mentioned previously.

The two points of view on renormalization lead to identical results, of course, since they are actually two different interpretations of the same procedure. In renormalizable theories, a *finite number* of counterterms are needed in order to render the theory UV finite. For the same purpose, in non-renormalizable theories we need an *infinite number* of counterterms.

7.6 Quantum corrections to scalar mass

Let us now discuss a very delicate point, quantum corrections to a scalar mass, again in the ϕ^4 theory. We now denote by m_0 the mass parameter in the lagrangian. We want to evaluate the quantum correction to the mass

$$m^2 \;=\; m_0^2 \;+\; i\,\Sigma\;, \tag{319}$$

where Σ is the (appropriately defined) quantum correction to the two-point Green function

$$\begin{aligned} G(p) \;&=\; D_F(p) \;+\; D_F(p)\,\Sigma\,D_F(p) + D_F\Sigma\,D_F(p)\Sigma\,D_F(p)\cdots \\ &= \frac{D_F}{1-\Sigma\,D_F} \;=\; \frac{i}{p^2-m_0^2-i\Sigma}\;. \end{aligned} \tag{320}$$

After the Wick rotation to euclidian momenta and at one-loop, Σ equals

$$\Sigma = \frac{1}{2}(-i\lambda)\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_0^2+i\epsilon}\;. \tag{321}$$

Let us evaluate the result with an euclidian momentum cutoff $0<p_E^2<\Lambda^2$. The result is

$$i\Sigma = \frac{\lambda}{32\pi^2}\int_0^{\Lambda^2} dp_E^2\frac{p_E^2}{p_E^2+m_0^2} = \frac{\lambda}{32\pi^2}\left(\Lambda^2 - m_0^2\ln\frac{\Lambda^2}{m_0^2}\right)\;. \tag{322}$$

The one-loop quantum-corrected mass is therefore

$$m^2 = m_0^2 + \frac{\lambda}{32\pi^2}\left(\Lambda^2 - m_0^2\ln\frac{\Lambda^2}{m_0^2}\right) \equiv\; m_0^2+\delta m^2\;. \tag{323}$$

Whereas the cutoff is unphysical for a fundamental theory, it is widely believed, in analogy with the cutoff interpretation in statistical mechanics, that a quantum computation is valid up to the energy scale where new physics appears. The mass of the scalar Higgs particle in the Standard Model was recently measured at LHC to be around 125 GeV. Since in the SM $m_h^2 \;=\; 2\lambda v^2$ and v is known, it means LHC measured the self-coupling λ. If the scale of new physics is Λ is very large, quantum corrections to the mass are also large. In the table below, we display the order of magnitude of quantum corrections to the scalar mass for a low and a high new physics scale Λ. If the quantum corrections are much larger than the physical mass, then there should be a delicate tuning between the bare mass m_0 and the quantum correction. Although there is no inconsistency if this would happen, this fine-tuning is considered to be unlikely to happen in nature.

Λ	10 TeV	10^{16} GeV
δm^2	(TeV)^2	10^{15} GeV

In case of theories where the electromagnetic, weak and strong interactions are unified at a high energy scale $\Lambda\sim 10^{16}$ GeV, the fine-tuning is of order 10^{-30} ! This is the so-called *high-hierarchy problem* of the Higgs mass in the Standard Model and is one of the main reasons to search for new physics in the TeV range.

Let us contrast this situation to the one of the electron (or any other fermion) mass in QED, of mass M. The quantum correction in QED with the electron/positron and the photon turns out to be given by

$$\delta M \;=\; \frac{3\alpha}{4\pi}M\ln\frac{\Lambda^2}{M^2}\;. \tag{324}$$

There is a major difference compared to the quantum correction to a scalar mass : the correction is proportional to the original mass itself and only logarithmically dependent on the cutoff. Even if the electron mass is tiny and even for a very large cutoff, the correction is always small in a perturbative theory. The tuning mentioned above for a scalar mass does not therefore exists for a fermion mass. The reason for this difference is that in the massless limit, the fermion acquires an additionl symmetry, the axial symmetry introduced in (85),(86). The axial symmetry forces quantum correction to vanish in the massless limit and protects the electron (fermion) mass from quantum corrections from any large mass scale. In this sense, small fermion masses are *natural*, whereas small scalar masses *are not* and lead to the hierarchy problem in the Standard Model.

7.7 Renormalization group and running of couplings

In concrete perturbation theory computations at n-loops for a field theory with coupling g, the result contain the appropriate factor α^n, where $\alpha = g^2/(4\pi)$, but actually there are logarithmic factors and there can be up to n logarithmic factors $\ln \frac{q^2}{m^2}$, where q is the typical momentum and m some physical mass. More precisely, there are of factors of the type

$$\left(\frac{\alpha}{4\pi} \ln \frac{q^2}{m^2} \right)^n \ . \tag{325}$$

Such factors invalide perturbation theory when $\frac{\alpha}{4\pi} \ln \frac{q^2}{m^2}$ is large, which can happen even if α is small for $q >> m$ or $q << m$. This problem also arises in massless theories like for example Yang-Mills theories, in which the mass m is replaced with some renormalization mass scale μ. The solution to such problems is the introduction of coupling constants $g(\mu)$ depending on the apriori arbitrary renormalization scale μ, chosen inteligently in order to minimize the effects of such logarithms, i.e. $\mu \sim E$, where E is the typical energy scale of the process under consideration. We can then do perturbation theory as long as $g(\mu)$ are small. In a certain precise sense, these large logarithms are resummed by defining the running couplings $g(\mu)$, which are commputable in a sense to be defined below.

Let us consider an n-point Green function that appears in computations of physical obervables

$$G^{(n)}(p_i, g, m, \mu) \ = \ Z^{\frac{n}{2}} \, G^{(n)}(p_i, g_0, m_0, \Lambda) \ . \tag{326}$$

The left-hand side contains the renormalized n-point Green function depending on the renormalization scale μ which is independent on the cutoff Λ. On the other hand, in the right-hand side sits the unrenormalized Green function depending on the cutoff Λ, which known nothing about the renormalization scale μ. In a renormalizable theory, the two are proportional, with a proportionality factor given by the appropriate power of the wave function renormalization Z. Since the unrenormalized function is independent on the sliding scale μ, one can write

$$\mu \frac{d}{d\mu} G^{(n)}(p_i, g_0, m_0, \Lambda) = 0 \to \left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \gamma_m(g) m \frac{\partial}{\partial m} + n\gamma(g) \right) G^{(n)}(p_i, g, m, \mu) = 0 \ , \tag{327}$$

where

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} \ , \ \gamma_m(g) = \mu \frac{\partial \ln m}{\partial \mu} \ , \ \gamma(g) = \frac{1}{2} \mu \frac{\partial \ln Z}{\partial \mu} \ . \tag{328}$$

The eqs. (327) satisfied by the renormalized Green functions are called Callan-Symanzik equations [39]. The role of the functions β, γ_m and γ will be transparent soon. In general they will also depend on ratios m/μ, but in some renormalization schemes (minimal subtraction) they depend only on the dimensionless couplings g. Let us now scale the external momenta $\partial_i \to \lambda p_i$, that we will use in order to study the asymptotic behaviour of the Green functions. If D is the canonical dimension of $G^{(n)}$, then by using Euler theorem for homogeneous functions one gets

$$\left(\lambda \frac{\partial}{\partial \lambda} + m \frac{\partial}{\partial m} + \mu \frac{\partial}{\partial \mu} - D \right) G^{(n)}(\lambda p_i, g, m, \mu) = 0 \ . \tag{329}$$

By combining (328) and (329), one obtains

$$\left(-\lambda\frac{\partial}{\partial\lambda}+\beta(g)\frac{\partial}{\partial g}-(1-\gamma_m(g))m\frac{\partial}{\partial m}+n\gamma(g)+D\right)G^{(n)}(\lambda p_i,g,m,\mu)=0\;. \tag{330}$$

Equation (330) allows to find the asymptotic behaviour of the Green functions. For this, one introduce the λ-dependent functions $\bar{g}(\lambda,g)$ and $\bar{m}(\lambda,g)$ satisfying the equations

$$\lambda\frac{\partial\bar{g}}{\partial\lambda}=\beta(\bar{g})\quad,\quad\lambda\frac{\partial\bar{m}}{\partial\lambda}=[\gamma_m(\bar{g})-1]\bar{m}\;. \tag{331}$$

and the boundary condition $\bar{g}(1,g)=g$, $\bar{m}(1,m)=m$. It can be shown that $\bar{g}$ and $\bar{m}$ satisfy the differential equations

$$\left[\lambda\frac{\partial}{\partial\lambda}-\beta(g)\frac{\partial}{\partial g}\right]\bar{g}(\lambda,g)=0\quad,\quad\left[\lambda\frac{\partial}{\partial\lambda}+(1-\gamma_m)m\frac{\partial}{\partial m}\right]\bar{m}(\lambda,m)=0\;. \tag{332}$$

With the help of these momentum-dependent couplings, the Green functions for rescaled momenta can be expressed as

$$G^{(n)}(\lambda p_i,g,m,\mu)\;=\;\lambda^D\,e^{n\int_0^{\ln\lambda}d\ln\lambda'\gamma(\bar{g})}\,G^{(n)}(p_i,\bar{g},\bar{m},\mu)\;. \tag{333}$$

The large logarithms are summed efficiently if, for momenta λp_i, the renormalization scale μ is chosen $\mu\sim\lambda E$. The "running" coupling $\bar{g}$ and masses $\bar{m}$ are obtained by consistently integrating out the RG eqs. (332) from the energy E characteristic of the process with external momenta p_i to the energy λE characteristic of the process with external momenta λp_i. They are therefore the appropriate quantities to be used, in order to (significantly) improve perturbation theory computations, for the processes with external momenta λp_i.

7.8 QED and the running of fine structure constant

We use here the *counterterm* method for the renormalization of QED. In this case, the initial lagrangian, the counterterms and their sum is

$$\begin{aligned}
\mathcal{L}\;&=\;-\frac{1}{4}F_{mn}^2+\bar{\Psi}(i\gamma^m\partial_m-q\gamma^mA_m-M)\Psi\;,\\
\delta\mathcal{L}\;&=\;-\frac{1}{4}(Z_3-1)F_{mn}^2+(Z_2-1)\bar{\Psi}i\gamma^m\partial_m\Psi\;,\\
&-(Z_1-1)q\bar{\Psi}\gamma^mA_m\Psi-(Z_M-1)M\bar{\Psi}\Psi\;,\\
\mathcal{L}_0\;&=\mathcal{L}+\delta\mathcal{L}=\;-\frac{1}{4}(F_{mn}^0)^2+\bar{\Psi}_0(i\gamma^m\partial_m-q_0\gamma^mA_m^0-M_0)\Psi_0\;.
\end{aligned} \tag{334}$$

The relations between the *bare and renormalized* quantities are then

$$\begin{aligned}
A_m^0\;&=\;Z_3^{1/2}\,A_m\quad,\quad\Psi_0\;=\;Z_2^{1/2}\,\Psi\;,\\
M_0\;&=\;\frac{Z_M}{Z_2}\,M\quad,\quad q_0\;=\;\frac{Z_1}{Z_2Z_3^{1/2}}\,q\;,
\end{aligned} \tag{335}$$

where Z_1 comes from the one-loop vertex correction, Z_2 (Z_3) is the fermionic (photon) wave function renormalization, whereas Z_M is the mass renormalization. In QED it can be shown that $Z_1=Z_2$, the so-called Ward identity. Then charge renormalization in QED comes only from vacuum polarization $q_0=Z_3^{-1/2}q$. The *RG running* can be found from

$$\mu\frac{\partial}{\partial\mu}q_0\;=\;0\quad\Rightarrow\quad\beta(q)\;=\;\mu\frac{\partial q}{\partial\mu}\;=\;q\frac{\partial\ln Z_3^{1/2}}{\partial\ln\mu}\;. \tag{336}$$

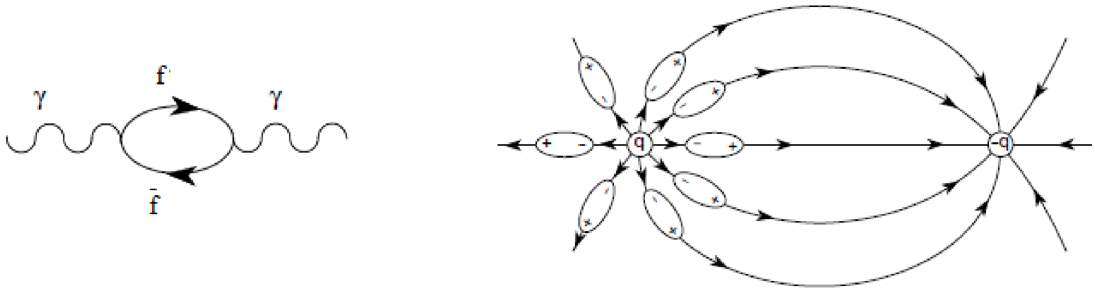


Fig. 16: Screening of electric charges by vacuum polarization provides an intuitive picture of the "running" of electric charge. Figure taken from [38].

By an explicit computation in QED with just the electron in the loop and by defining the fine-structure constant $\alpha = q^2/(4\pi)$, we find

$$Z_3 \; = \; 1 - \frac{\alpha}{3\pi} \ln \frac{\Lambda}{\mu} \; + \; \text{finite} \; , \tag{337}$$

where μ is an arbitrary, renormalization scale. We then find

$$\beta(q) \; = \; \frac{q^3}{24\pi^2} \quad \Rightarrow \quad \frac{1}{\alpha(Q)} = \frac{1}{\alpha(\mu)} - \frac{1}{3\pi} \ln \frac{Q}{\mu} \; . \tag{338}$$

We found therefore that the fine structure coupling *increases with energy* ! This can be intuitively interpreted due to the screening of the electric charge by electron-positron pairs from the quantum vacuum (see Figure 16). Experimentally, if at low energies $\alpha(0) \sim 1/137$, it was measured, in the sense described above, that the running electromagnetic coupling for energiesof the order of the M_Z mass is indeed larger, $\alpha(M_Z) \sim 1/128$.

The situation for the strong coupling α_3 is different due to the non-abelian nature of the interaction. The result is an *anti-screening* due to gluon self-interactions [40].

There is a tantalizing hint of unification of gauge couplings at high-energy, as seen from figure 17, that could point towards a unified gauge structure at high-energy [41]. Running couplings and renormalization are important everywhere in the SM and its applications. For example :
• In any process the couplings have to be evaluated at the relevant energy scale. Ex :
- In $\pi^0 \to \gamma\gamma$, the fine-structure constant has to be evaluated at the pion mass $\alpha(m_\pi)$.
- Identification of relevant momenta and RGE of operators in QCD is crucial in order to extrapolate perturbative quantities down in energy via the renormalization group.
• In the study of electroweak baryogenesis, for example the scalar potential

$$V(\Phi) \; \simeq \; - \, \mu^2(\langle\Phi\rangle|)|\Phi|^2 \; + \; \lambda(\langle\Phi\rangle) \, |\Phi|^4 \tag{339}$$

is to be evaluated at the minimum at the scalar potential.

8 Global and gauge anomalies

Let us consider a Dirac fermion coupled to a $U(1)$ gauge field

$$\mathcal{L} = \bar{\Psi} i \gamma^m D_m \Psi - M \bar{\Psi}\Psi \; . \tag{340}$$

In the massless limit $M \to 0$, the model has a vector and an axial symmetry $U(1)_V \times U(1)_A$. The corresponding Noether currents

$$J_m \; = \; \bar{\Psi}\gamma_m\Psi \quad , \quad J_m^5 \; = \; \bar{\Psi}\gamma_m\gamma_5\Psi \tag{341}$$

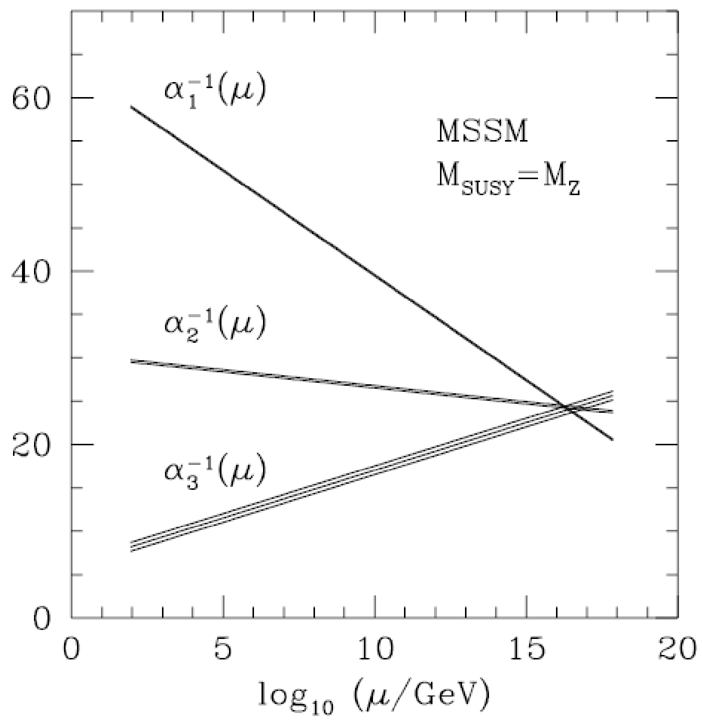


Fig. 17: Extrapolation of gauge couplings in the (minimal supersymmetric extension) of the Standard Model (figure taken from [42]) : hint of unification of couplings at high energy ?

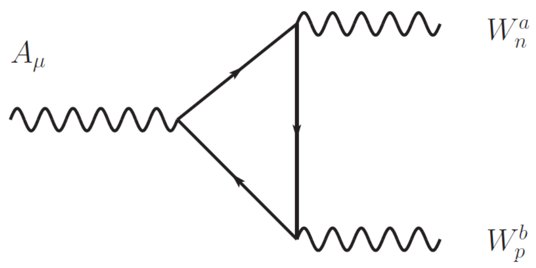


Fig. 18: Adler-Bell-Jackiw triangle anomalies.

satisfy classically

$$\partial^m J_m = 0 \quad , \quad \partial^m J_m^5 = 2iM\bar{\Psi}\gamma_5\Psi \ . \tag{342}$$

At the quantum level, these conservation laws are modified. It was shown in [24] that even in the massless limit it is not possible to preserve simultaneously the vector and the axial symmetry, due to subtleties coming from triangle graphs, see fig. (18).

8.1 Triangle anomalies: the general computation

In what follows we follow closely [6] and we study the quantum anomaly in a system of left-handed and right-handed fermions coupled to gauge fields via the current

$$J_a^\mu = \bar{\psi}_L t_a^L \gamma^\mu \psi_L + \bar{\psi}_R t_a^R \gamma^\mu \psi_R \ . \tag{343}$$

We will use a basis for the fermions so that they are all left-handed. In four dimensions this can always be done with the help of the charge-conjugation matrix C. Indeed, if χ is a Dirac fermion, $\frac{1-\gamma_5}{2}\chi$ projects into left-handed components in the representation $(1/2, 0)$ of the Lorentz group, whereas $\frac{1+\gamma_5}{2}\chi$ projects into right-handed components in the Lorentz group representation $(0, 1/2)$. We will package both of them into a single spinor ψ with all components into the representation $(1/2, 0)$, which is of the type

$$\psi = \begin{pmatrix} \frac{1-\gamma_5}{2}\chi \\ \frac{1-\gamma_5}{2}\chi^c \end{pmatrix} \tag{344}$$

where $\chi^c = C\bar{\chi}^T$ is the charge-conjugated spinor.

The associated generator (charge operator for $U(1)$ symmetries) for the gauge field A_μ^a is denoted by T_a. We define the various symmetry currents as

$$J_a^\mu = \bar{\psi} T_a \gamma^\mu \psi \tag{345}$$

where the symmetry generator T_a acts on the packaged left-handed spinor ψ as the matrix block

$$T_a = \begin{pmatrix} t_a^L & 0 \\ 0 & -(t_a^R)^\star \end{pmatrix} = \begin{pmatrix} t_a^L & 0 \\ 0 & -(t_a^R)^T \end{pmatrix} , \tag{346}$$

where for a given Dirac fermion χ the symmetry generators are defined through the infinitesimal transformation

$$\delta\chi = i\alpha_a \left[\frac{1-\gamma_5}{2} t_a^L + \frac{1+\gamma_5}{2} t_a^R\right] \chi \, . \tag{347}$$

The three-current correlator we will study is

$$\Gamma_{abc}^{\mu\nu\rho}(x,y,z) \;=\; \langle 0|T(J_a^\mu(x) J_b^\nu(y) J_c^\rho(z))|0\rangle \, . \tag{348}$$

For conserved currents, the naive Ward identity for the divergence of such a correlator is

$$\partial_\mu \Gamma_{abc}^{\mu\nu\rho}(x,y,z) = +if_{abd}\delta^4(x-y)\langle 0|T(J_d^\nu(y) J_c^\rho(z))|0\rangle + if_{acd}\delta^4(x-y)\langle 0|T(J_b^\nu(y) J_d^\rho(z))|0\rangle \, , \tag{349}$$

where f_{abc} are the group structure constants. The leading contribution at one loop emerges from fermions going around the loop. The total contribution is obtained by summing over all relevant fermion fields.

There are two diagrams for the correlator that can be evaluated to yield

$$\begin{aligned} \Gamma_{abc}^{\mu\nu\rho}(x,y,z) \;=\; & \mathrm{Tr}\left[S_F(x-y) T_b \gamma^\nu P_L S_F(y-z) T_c \gamma^\rho P_L S_F(z-x) T_a \gamma^\mu P_L\right] \\ & + \mathrm{Tr}\left[S_F(x-z) T_c \gamma^\rho P_L S_F(z-y) T_b \gamma^\nu P_L S_F(y-x) T_a \gamma^\mu P_L\right] \end{aligned} \tag{350}$$

with

$$P_L = \frac{1-\gamma^5}{2} \quad , \quad S_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{i\not{p}}{p^2 + i\epsilon} e^{ip\cdot x} \, . \tag{351}$$

Substituting one obtains

$$\begin{aligned} \Gamma_{abc}^{\mu\nu\rho}(x,y,z) \;=\; & -i \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} e^{-i(k_1+k_2)\cdot x + ik_1\cdot y + ik_2\cdot z} \int \frac{d^4p}{(2\pi)^4} \times \\ & \Bigg\{ \mathrm{Tr}\left[\frac{\not{p} - \not{k}_1 + \not{a}_1}{(p-k_1+a_1)^2 + i\epsilon} \gamma^\nu \frac{\not{p} + \not{a}_1}{(p+a_1)^2 + i\epsilon} \gamma^\rho \frac{\not{p} + \not{k}_2 + \not{a}_1}{(p+k_2+a_1)^2 + i\epsilon} \gamma^\mu P_L\right] \mathrm{tr}[T_b T_c T_a] + \\ & + \mathrm{Tr}\left[\frac{\not{p} - \not{k}_2 + \not{a}_2}{(p-k_2+a_2)^2 + i\epsilon} \gamma^\rho \frac{\not{p} + \not{a}_2}{(p+a_2)^2 + i\epsilon} \gamma^\nu \frac{\not{p} + \not{k}_1 + \not{a}_2}{(p+k_1+a_2)^2 + i\epsilon} \gamma^\mu P_L\right] \mathrm{tr}[T_c T_b T_a] \Bigg\} \, . \end{aligned} \tag{352}$$

We have shifted the integrated momentum in the two diagrams using two vectors $a_{1,\mu}$ and $a_{2,\mu}$. This reflects an ambiguity of the triangle graph, even if the integrals are actually convergent, and translates into the definition of the associated current operators, corresponding to a certain freedom to move the anomaly from one current to another. By using the identities

$$\not{k}_1 + \not{k}_2 = (\not{p} + \not{k}_2 + \not{a}_1) - (\not{p} - \not{k}_1 + \not{a}_1) = (\not{p} + \not{k}_1 + \not{a}_2) - (\not{p} - \not{k}_2 + \not{a}_2) \,, \tag{353}$$

one finds

$$\begin{aligned} \partial_\mu \Gamma^{\mu\nu\rho}_{abc}(x,y,z) \;=\; & -\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} e^{-i(k_1+k_2)\cdot x + ik_1\cdot y + ik_2\cdot z} \int \frac{d^4p}{(2\pi)^4} \times \\ & \Big\{ \mathrm{tr}[T_bT_cT_a]\mathrm{Tr}\left[\frac{\not{p} - \not{k}_1 + \not{a}_1}{(p-k_1+a_1)^2 + i\epsilon}\gamma^\nu \frac{\not{p} + \not{a}_1}{(p+a_1)^2 + i\epsilon}\gamma^\rho P_L\right] \\ & -\mathrm{tr}[T_cT_bT_a]\mathrm{Tr}\left[\frac{\not{p} + \not{a}_2}{(p+a_2)^2 + i\epsilon}\gamma^\nu \frac{\not{p} + \not{k}_1 + \not{a}_2}{(p+k_1+a_2)^2 + i\epsilon}\gamma^\rho P_L\right] \\ & +\mathrm{tr}[T_cT_bT_a]\mathrm{Tr}\left[\frac{\not{p} - \not{k}_2 + \not{a}_2}{(p-k_2+a_2)^2 + i\epsilon}\gamma^\rho \frac{\not{p} + \not{a}_2}{(p+a_2)^2 + i\epsilon}\gamma^\nu P_L\right] \\ & -\mathrm{tr}[T_bT_cT_a]\mathrm{Tr}\left[\frac{\not{p} + \not{a}_1}{(p+a_1)^2 + i\epsilon}\gamma^\rho \frac{\not{p} + \not{k}_2 + \not{a}_1}{(p+k_2+a_1)^2 + i\epsilon}\gamma^\nu P_L\right]\Big\} \,. \end{aligned} \tag{354}$$

It is now convenient to separate the group theory trace into a symmetric and antisymmetric part

$$\begin{aligned} \mathrm{tr}[T_bT_cT_a] &= d_{abc} + \frac{i}{2}Nf_{abc} \,, \\ \mathrm{tr}[T_cT_bT_a] &= d_{abc} - \frac{i}{2}Nf_{abc} \,, \end{aligned} \tag{355}$$

where $d_{abc} = \frac{1}{2}\mathrm{tr}[\{T_a, T_b\}T_c]$ is totally symmetric in abc. The contribution of the term proportional to the group structure constants f_{abc} precisely reproduce the naive Ward identity (349) and has nothing to do with the quantum anomaly. The remaining, symmetric part, is

$$\begin{aligned} \partial_\mu \Gamma^{\mu\nu\rho}_{abc}(x,y,z) \;=\; & -d_{abc}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} e^{-i(k_1+k_2)\cdot x + ik_1\cdot y + ik_2\cdot z} \int \frac{d^4p}{(2\pi)^4} \times \\ & \Big\{ \mathrm{Tr}\left[\frac{\not{p} - \not{k}_1 + \not{a}_1}{(p-k_1+a_1)^2 + i\epsilon}\gamma^\nu \frac{\not{p} + \not{a}_1}{(p+a_1)^2 + i\epsilon}\gamma^\rho P_L\right] \\ & -\mathrm{Tr}\left[\frac{\not{p} + \not{a}_2}{(p+a_2)^2 + i\epsilon}\gamma^\nu \frac{\not{p} + \not{k}_1 + \not{a}_2}{(p+k_1+a_2)^2 + i\epsilon}\gamma^\rho P_L\right] \\ & +\mathrm{Tr}\left[\frac{\not{p} - \not{k}_2 + \not{a}_2}{(p-k_2+a_2)^2 + i\epsilon}\gamma^\rho \frac{\not{p} + \not{a}_2}{(p+a_2)^2 + i\epsilon}\gamma^\nu P_L\right] \\ & -\mathrm{Tr}\left[\frac{\not{p} + \not{a}_1}{(p+a_1)^2 + i\epsilon}\gamma^\rho \frac{\not{p} + \not{k}_2 + \not{a}_1}{(p+k_2+a_1)^2 + i\epsilon}\gamma^\nu P_L\right]\Big\} \,. \end{aligned} \tag{356}$$

Grouping together the first two and the last two trace factors, one arrives at

$$\begin{aligned} & \partial_\mu \Gamma^{\mu\nu\rho}_{abc}(x,y,z) = -d_{abc}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} e^{-i(k_1+k_2)\cdot x + ik_1\cdot y + ik_2\cdot z} \int \frac{d^4p}{(2\pi)^4} \times \\ & \Big\{ \mathrm{Tr}[\gamma^k\gamma^\nu\gamma^\lambda\gamma^\rho \frac{1-\gamma_5}{2}]\, I_{k\lambda}(a_1 - a_2 - k_1, a_2, a_2 + k_1) \\ & +\mathrm{Tr}[\gamma^k\gamma^\rho\gamma^\lambda\gamma^\nu \frac{1-\gamma_5}{2}]\, I_{k\lambda}(a_2 - a_1 - k_2, a_1, a_1 + k_2)\Big\} \,, \end{aligned} \tag{357}$$

where

$$I_{k,\lambda}(k,c,d) = \int \frac{d^4p}{(2\pi)^4}\, [f_{k,\lambda}(p+k,c,d) - f_{k,\lambda}(p,c,d)] \,,$$

$$\text{and} \quad f_{k,\lambda}(p,c,d) \;=\; \frac{(p+c)_k(p+d)_\lambda}{[(p+c)^2+i\epsilon][(p+d)^2+i\epsilon]}\;. \tag{358}$$

These integrals can be computed by Taylor expansion in powers of k. It can be shown that each contribution in the Taylor expansion is a surface integral and that only the terms linear and quadratic in k in the expansion do contribute. After an explicit computation, the result is

$$I_{k,\lambda}(k,c,d) = \frac{i}{96\pi^2}\left[2k_\lambda c_k + 2k_k d_\lambda - k_\lambda d_k - k_k c_\lambda - \eta_{k\lambda} k(k+c+d)\right]\;. \tag{359}$$

Demanding that there is no anomaly in the vector currents implies that the term without γ_5 in (357) vanishes. Due to the symmetry of this term in k,λ and ν,ρ, this term appears in the combination

$$\begin{aligned}&I_{k,\lambda}(a_1-a_2-k_1,a_2,a_2+k_1)+I_{\lambda,k}(a_1-a_2-k_1,a_2,a_2+k_1)\\ &+I_{k,\lambda}(a_2-a_1-k_2,a_1,a_1+k_2)+I_{\lambda,k}(a_2-a_1-k_2,a_1,a_1+k_2)\;.\end{aligned} \tag{360}$$

This, and actually also the anomalies in the other currents, vanish for $a_2 = -a_1$, choice that we keep from now on. The vector a_1 is therefore parameterizing the leftover scheme dependence of the triangle graph in question.

We are left with the term in the trace containing γ_5, for which we use

$$\mathrm{Tr}[\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\tau\gamma_5] = -4i\,\epsilon^{\nu\rho\sigma\tau}\;. \tag{361}$$

By using (361), we obtain the following divergence formulae

$$\partial_\mu\Gamma^{\mu\nu\rho}_{abc}(x,y,z) = -\frac{d_{abc}}{8\pi^2}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}e^{-i(k_1+k_2)\cdot x+ik_1\cdot y+ik_2\cdot z}\;\epsilon^{\nu\rho\sigma\tau}\;a_{1,\sigma}(k_1+k_2)_\tau \tag{362}$$

$$\partial_\nu\Gamma^{\mu\nu\rho}_{abc}(x,y,z) = -\frac{d_{abc}}{8\pi^2}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}e^{-i(k_1+k_2)\cdot x+ik_1\cdot y+ik_2\cdot z}\;\epsilon^{\mu\rho\sigma\tau}\;(a_1+k_2)_\sigma(k_1)_\tau \tag{363}$$

$$\partial_\rho\Gamma^{\mu\nu\rho}_{abc}(x,y,z) = -\frac{d_{abc}}{8\pi^2}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}e^{-i(k_1+k_2)\cdot x+ik_1\cdot y+ik_2\cdot z}\;\epsilon^{\mu\nu\sigma\tau}\;(k_1-a_1)_\sigma(k_2)_\tau \tag{364}$$

A close investigation of the anomalous structure of the three currents (364) shows that it is not possible to choose the shift vector a_1 in order to eliminate the anomaly altogether. The choice $a_1 \sim k_1+k_2$ eliminates the anomaly from the current J_a, whereas $a_1 \sim \pm k_1 - k_2$ ($a_1 \sim k_1 \pm k_2$) eliminates the anomaly from the current J_b (J_c). For generic vectors k_1, k_2 there is no common solution to these three anomalies. A generic choice of scheme (i.e. $a_{1,\mu}$) indicates that the divergence structure is asymmetric among the three vertices of the triangle graph. The choice should be dictated by physical requirements. In the class investigated in the next subsection, J_a is the current of a global symmetry, whereas J_b and J_c are currents of gauge symmetries, coupling to gauge fields. In this case, one has to choose a_1 so that only J_a has an anomaly, which has the unique solution

$$a_1 = k_1 - k_2\;. \tag{365}$$

In this case, the anomaly in the current J_a becomes

$$\partial_\mu\Gamma^{\mu\nu\rho}_{abc}(x,y,z) = \frac{1}{4\pi^2}d_{abc}\epsilon^{\alpha\beta\nu\rho}\frac{\partial\delta^4(y-x)}{\partial y^\alpha}\frac{\partial\delta^4(z-x)}{\partial z^\beta}\;. \tag{366}$$

One can interpret this result as a quantum contribution to the current in the presence of gauge fields

$$\langle J^\mu_a\rangle_q = \frac{g_b g_c}{2}\int d^4x d^4y \Gamma^{\mu\nu\rho}_{abc}(x,y,z)A^b_\nu A^c_\rho\;, \tag{367}$$

leading to the anomalous divergence

$$\langle \partial_\mu J_a^\mu \rangle_{\rm an} = \frac{g_b g_c}{8\pi^2} d_{abc} \epsilon^{\alpha\nu\beta\rho} \partial_\alpha A_\nu^b \partial_\beta A_\rho^c \ . \tag{368}$$

In the non-abelian gauge fields case, there are additional contributions from square and pentagon diagrams. In both the abelian and non-abelian case, one obtains the gauge-invariant result

$$\langle \partial_\mu J_a^\mu \rangle_{\rm an} = \frac{g_b g_c}{32\pi^2} d_{abc} \epsilon^{\alpha\nu\beta\rho} F_{\alpha\nu}^b F_{\beta\rho}^c \ . \tag{369}$$

Notice that the explicit form of the group-theory coefficient in terms of left and right-handed original fermions is

$$d_{abc} \ = \ \frac{1}{2}{\rm Tr}[\{t_a^L, t_b^L\} t_c^L] - \frac{1}{2}{\rm Tr}[\{t_a^R, t_b^R\} t_c^R] \ . \tag{370}$$

We have therefore shown that Symmetries of the classical action can have *anomalies* at the quantum level, generated by one-loop *triangle diagrams*. There are two different cases to consider:
- anomalies in the conservation of a global symmetry current
- anomalies for gauged symmetries.

8.2 Global anomalies

For global symmetries, quantum anomalies do not creates consistency problems; they actually play an important role in QCD in the so-called η' problem and in the electromagnetic decay of the pion $\pi^0 \to \gamma\gamma$. For a global symmetry with Noether current J_m^a of generator T^a, the anomaly in operatorial form is given by

$$\begin{aligned} \partial^m J_m^a &= -\frac{g^2}{16\pi^2} \epsilon^{mnpq} \, tr(T^a F_{mn} \, F_{pq}) \ , \\ \text{where} \quad tr(T^a F_{mn} \, F_{pq}) \ &= \ \frac{1}{2} \, tr(T^a \{T^A, T^B\}) \, F_{mn}^A \, F_{pq}^B \ , \end{aligned} \tag{371}$$

where g is the gauge coupling of the gauge group with corresponding gauge fields $A_m = A_m^A T^A$. In (371) the trace is computed over all the fermionic spectrum of the theory, considered in this section to be of Dirac type, i.e. $t^L = -t^R$ in eq. (370). Let us consider to start with a Dirac fermion coupled to a $U(1)$ gauge field

$$\mathcal{L} = \bar\Psi i \gamma^m D_m \Psi - M \bar\Psi \Psi \ . \tag{372}$$

In the massless limit $M \to 0$, the model has a vector and an axial symmetry $U(1)_V \times U(1)_A$. The corresponding Noether currents

$$J_m \ = \ \bar\Psi \gamma_m \Psi \quad , \quad J_m^5 \ = \ \bar\Psi \gamma_m \gamma_5 \Psi \tag{373}$$

satisfy

$$\partial^m J_m = 0 \quad , \quad \partial^m J_m^5 = 2iM \bar\Psi \gamma_5 \Psi - \frac{e^2}{16\pi^2} \epsilon^{mnpq} \, F_{mn} \, F_{pq} \ , \tag{374}$$

where the last term in the second divergence in (374) is the *quantum anomaly*. Even if the vector and the axial currents in (374) are both classically conserved in the massless limit $M = 0$, there is *no regularization* preserving both the conservation of the vector and of the axial current, as shown in the previous section. If $U(1)_V$ is a gauge symmetry (like the gauge symmetry in electromagnetism), we have for consistency of the theory to choose a regularization preserving the vector current conservation. As a consequence, one is forced to accept the existence of an anomaly which violates at the quantum level the axial current conservation. This explains actually why the η' meson is not a pseudo-Goldstone boson for

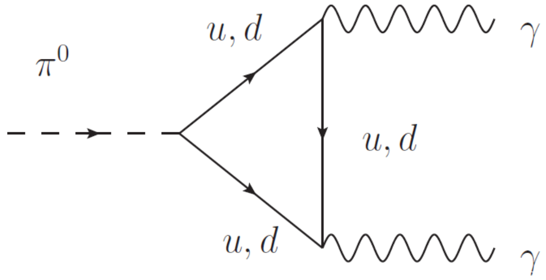


Fig. 19: The electromagnetic pion desintegration $\pi^0 \to \gamma\gamma$ is related to the axial $U(1)_A$ anomaly.

the dynamical chiral symmetry breaking $U(2)_L \times U(2)_R = SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_A \Rightarrow SU(2)_V \times U(1)_B$ in QCD. Indeed, in this case the $U(1)_A$ axial current has the QCD anomaly

$$J_m^{U(1)_A} = \bar{u}\gamma_m\gamma_5 u + \bar{d}\gamma_m\gamma_5 d \, ,$$
$$\partial^m J_m^{U(1)_A} \;=\; 2i(m_u \bar{u}\gamma_5 u + m_d \bar{d}\gamma_5 d) - \frac{g_3^2}{16\pi^2}\epsilon^{mnpq}\, F_{mn}^A\, F_{pq}^A \, , \tag{375}$$

where F^A is the gluon field strength and g_3 is the color $SU(3)_c$ gauge coupling . Due to the explicit breaking of the axial symmetry by quark masses and the nonperturbative instanton effects, the η' gets a mass larger than the pions $\pi^\pm, \pi^0$, which are the pseudo-goldstone bosons of the axial $SU(2)_A$ symmetry. Another manifestation of the axial anomaly is in the electromagnetic pion decay $\pi^0 \to \gamma\gamma$. Let us define the $SU(2)$ currents

$$J_m^a = \bar{q}\gamma_m \frac{\tau^a}{2} q \quad , \quad J_m^{5,a} = \bar{q}\gamma_m\gamma_5 \frac{\tau^a}{2} q \, , \tag{376}$$

where $q = (u, d)^T$ and τ^a are the Pauli matrices. The fact that the pions are *Goldstone bosons* of the axial $SU(2)_A$ implies that the corresponding currents have a non vanishing matrix element between the vacuum and a one pion state

$$\langle 0 | J_m^{5,a}(x) | \pi^b(p) \rangle \;=\; i\, p_m\, f_\pi\, \delta^{ab}\, e^{-ipx} \, , \tag{377}$$

where the mass parameter f_π is called the pion decay constant. The axial isospin currents have no QCD anomalies since $tr(\tau^a\{T^A, T^B\}) = 0$ as a result of the isospin symmetry of strong interactions, but $J_m^{5,a}$ have an *anomaly* from the electromagnetic coupling

$$\partial^m J_m^{5,3} \;=\; -\frac{1}{16\pi^2}\epsilon^{mnpq}\, F_{mn}\, F_{pq}\, tr(Q^2 \frac{\tau_3}{2}) \;=\; -\frac{N_c e^2}{96\pi^2}\epsilon^{mnpq}\, F_{mn}\, F_{pq} \, , \tag{378}$$

where $Q = diag\ (2e/3, -e/3)$ is the matrix of the quark electric charges $q_u = 2e/3$, $q_d = -e/3$ and $N_c = 3$ is the number of quark colors.

By using (377) and (378) and using that under an axial $SU(2)_A$ with quarks $q = (u, d)$ transforming as $\delta q = i\gamma_5 \frac{\tau_3}{2} q$, the pion transforms like a Goldstone boson $\delta\pi^0 = \alpha f_\pi$, we obtain that the effect of the anomaly is to generate an effective pion-photon-photon coupling

$$\mathcal{L}_{\text{eff}} \;=\; \frac{\pi^0}{f_\pi}\partial^m J_m^{5,3} \;=\; -\frac{N_c e^2}{96\pi^2 f_\pi}\pi^0 \epsilon^{mnpq}\, F_{mn}\, F_{pq} \, . \tag{379}$$

In order to obtain (379), we have used the Noether theorem: the variation of the lagrangian under a transformation generated by the parameter α is equal to $\delta\mathcal{L} = \alpha\ \partial^m J_m$, where J_m is the corresponding Noether current. The effective lagrangian (379) is its low-energy manifestation; its variation under the axial transformations precisely repoduces the anomaly of the microscopic lagrangian. Using this effective coupling, the amplitude of the pion decay is computed to be

$$\mathcal{M}(\pi^0 \to \gamma\gamma) \ = \ - \frac{\alpha}{\pi f_\pi} \epsilon^{mnrs} \epsilon_n^* \epsilon_s^* p_m k_r \ , \tag{380}$$

where (p, ϵ_n) and (k, ϵ_s) are the momenta and polarization of the two photons. By summing over the photon polarizations

$$\sum_{\rm pol.} |\epsilon^{mnrs} \epsilon_n^* \epsilon_s^* p_m k_r|^2 \ = \ 2\, (pk)^2 \ = \ \frac{m_\pi^4}{2} \ , \tag{381}$$

we finally obtain the pion decay width

$$\Gamma(\pi^0 \to \gamma\gamma) \ = \frac{1}{32 \pi m_\pi} \sum_{\rm pol.} |\mathcal{M}(\pi^0 \to \gamma\gamma)|^2 \ = \ \frac{\alpha^2}{64 \pi^3} \frac{m_\pi^3}{f_\pi^2} \ , \tag{382}$$

which is in excellent agreement with the experimental branching ratio of the pion decay into two photons $\Gamma(\pi^0 \to \gamma\gamma) = (1.19 \pm 0.08) \times 10^{16} s^{-1}$.

Another interesting application of quantum anomalies is to the so-called *strong CP problem*. There is no symmetry principle forbidding the following term in the QCD lagrangian

$$\mathcal{L}_\theta \ = \ \theta \, \frac{g^2}{32 \pi^2} \, \epsilon^{mnpq} \, Tr(F_{mn} \, F_{pq}) \ , \tag{383}$$

where θ is a real (angular) parameter. Even if this can be shown to be a total derivative, topological (instanton) configurations in QCD makes this term to have nontrivial consequences. Actually, even if for some reason the original theta parameter is zero in the QCD lagrangian, it will be generated from the unitary redefinitions (241) that we were forced to perform in order to diagonalize the quark mass matrices. Indeed, the $U(1)_A$ part of these transformations is anomalous, leading to a change in the theta parameter

$$\theta \ \to \ \theta \ - \ \frac{1}{2} \, arg \ det \ m^q \ , \tag{384}$$

where $det \ m^q = det \ m^u \ det \ m^d$ is the product of the quark mass matrices. The theta parameter *violates the CP symmetry of strong interactions and the gluonic term generates a neutron dipole moment of order* $d_n \ \sim \ |\theta| \ \frac{e m_\pi^2}{m_N^3} \ \sim \ 10^{-16} |\theta| \ ecm$*, in conflict with the experimental data unless* $\theta \ < \ 10^{-10}$. This leads to the so-called *strong CP problem*. The problem would be absent if the up-quark mass would be zero, since in this case the theta parameter could be shifted to zero by an up-quark chiral redefinition. The masslessness of the up-quark is however probably excluded by now. The (commonly considered as the) most elegant solution to the strong CP problem is by postulating the existence of a new particle, *the axion* a [25]. If :

- there is a new abelian $U(1)_{PQ}$, *spontaneously broken global symmetry*, with the corresponding pseudo-Goldstone boson a called *the axion*, and with the symmetry breaking scale f_a
- such that $U(1)_{PQ}$ has triangle anomalies with the QCD gauge group $U(1)_{PQ} SU(3)_c^2$,

then *the anomaly* generates *new couplings* in the effective lagrangian which shift the θ parameter

$$\frac{g_3^2}{32 \pi^2} \, \xi \frac{a(x)}{f_a} \, \epsilon^{mnpq} \, Tr(F_{mn} \, F_{pq}) \ \Rightarrow \ \theta_{\rm eff} = \theta + \xi \frac{a}{f_a} \ , \tag{385}$$

where ξ is a model-dependent parameter parametrizing the strength of the axion couplings to matter. The θ parameter becomes therefore a dynamical quantity that will be dynamically determined by the minimization of the scalar potential of the axion. On the other hand, *non-perturbative QCD instanton effects* generate an axion potential of the type

$$V(a) \ \sim \ \frac{g_3^2}{32 \pi^2} \Lambda_{QCD}^4 \, \left[1 - \cos\, (\xi \frac{a(x)}{f_a} + \theta) \right] \ . \tag{386}$$

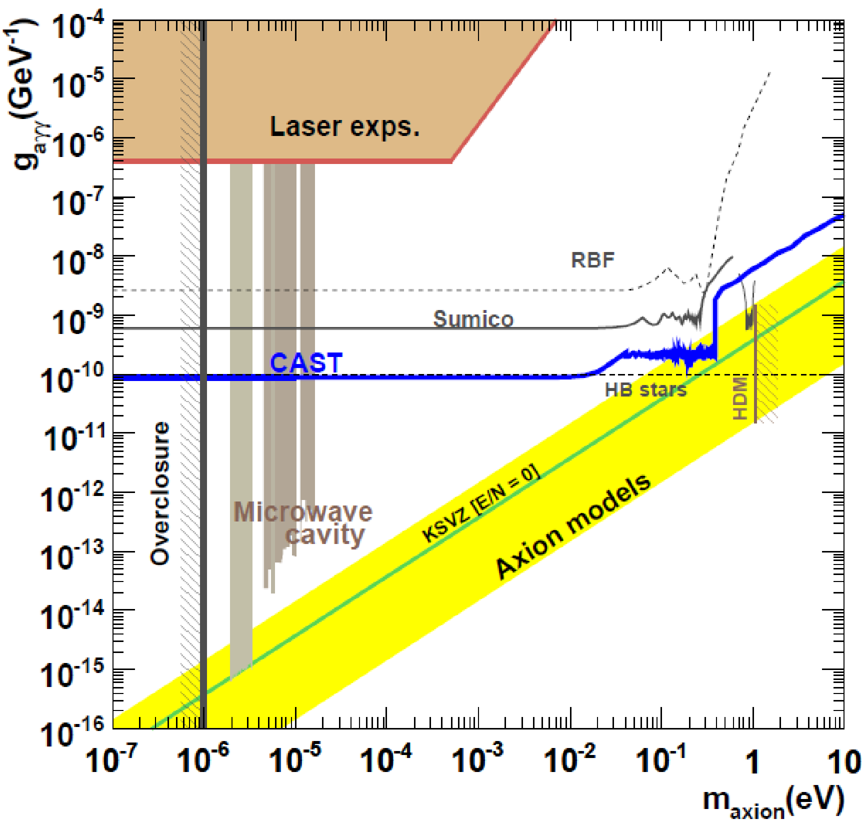


Fig. 20: Axion searches and constraints in the plane $g_{a\gamma}$ (the axion-photon coupling) versus the axion mass. See for example [43] for more details.

The minimum of the scalar potential is then reached at

$$\theta_{\text{eff}} \;=\; 0 \qquad \text{and the axion mass is } \; m_a \;\sim\; \xi \frac{g_3}{4\sqrt{2}\pi} \frac{\Lambda^2_{QCD}}{f_a} \;. \tag{387}$$

Axions were intensively searched since the 80's. They have particular couplings to gauge fields and to fermions. In particular, in the presence of electromagnetic fields, they can convert into photons. We display in fig. 20 some recent constraints in the plane $(g_{a\gamma}, m_a)$, where $g_{a\gamma}$ is the axion-photon coupling. On the theoretical side, axions are also present in *most SUSY and extensions* of the SM and play a crucial role in string theories, from various point of view, as e.g. anomaly cancelation and stabilization of moduli fields describing the geometry of the internal space.

Let us finish this part with a comment. The quantum anomaly is actually a total derivative :

$$\epsilon^{mnpq}\, Tr(F_{mn}\, F_{pq}) \;=\; \partial^m\, K_m \;, \tag{388}$$

where

$$K_\mu = 2\epsilon_{\mu\nu\alpha\beta}\left(A^{\nu A}\partial^\alpha A^{\beta A} + \frac{g}{3} f^{ABC} A^{\nu A} A^{\alpha B} A^{\beta C}\right) \;=\; \epsilon_{\mu\nu\alpha\beta}\left(A^{\nu A} F^{\alpha\beta A} - \frac{g}{3} f^{ABC} A^{\nu A} A^{\alpha B} A^{\beta C}\right) . \tag{389}$$

Despite this, in the non-abelian case, classical Yang-Mills configurations generate highly non-trivial effects like the appearance of *the theta angle in QCD and of the baryon (B) and lepton (L) number non-conservation*. Indeed, the *baryonic and leptonic currents* for example have an $U(1)_B \times SU(2)^2_L$ anomaly

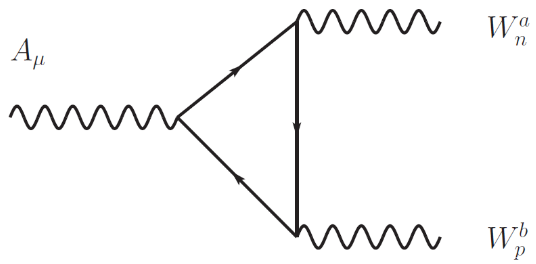


Fig. 21: One-loop gauge anomalies, if present, render the theory *inconsistent* at the quantum level.

they (have also an $U(1)_B \times U(1)_Y^2$ anomaly, which however plays no role in what follows)

$$J_m^B = \frac{1}{3}\sum_i (\bar{q}_i \gamma_m q_i + \bar{u}_{iR}\gamma_m u_{iR} + \bar{d}_{iR}\gamma_m d_{iR}) \ , \ \partial^m J_m^B = N_f \times \frac{g^2}{16\pi^2}\epsilon^{mnpq}\, Tr(F_{mn}\, F_{pq}) \ ,$$
$$J_m^L = \sum_i (\bar{l}_i \gamma_m l_i + \bar{e}_{iR}\gamma_m e_{iR}) \ , \ \partial^m J_m^B = \partial^m J_m^L \ , \tag{390}$$

where $N_f = 3$ is the number of families, leading to processes with a change in the baryon number

$$\Delta B = B_f - B_i = \int d^4x \partial^m J_m^B = \frac{N_f g^2}{16\pi^2}\int d^4x \epsilon^{mnpq} \ Tr(F_{mn}\, F_{pq}) \equiv N_f N_{\rm CS} \ ,$$
$$\text{where} \quad N_{\rm CS} = \frac{g^2}{16\pi^2}\oint d\Sigma^\mu \, K_\mu \tag{391}$$

is the so-called Chern-Simons number. Even if the result is a total derivative, the baryon number is violated by classical gauge field configurations *vanishing slowly* at infinity. Such configurations are characterized by zero field strength at the infinity $W_{mn} = 0 \leftrightarrow W_m = (i/g) U \partial_m U^{-1}$. The Chern-Simons number can then be expressed as

$$N_{\rm CS} = \frac{1}{12\pi^2}\int d^3x \epsilon^{ijk} Tr(U\partial_i U^{-1} U \partial_j U^{-1} U \partial_k U^{-1}) \ . \tag{392}$$

For $SU(2)$ gauge fields, such configurations are classified by gauge transformations $U \ : \ S^3 \to SU(2)$ and define the third homotopy group $\Pi_3(SU(2)) = Z$. $N_{\rm CS}$ is therefore restricted to integer values and $\Delta B = N_f \times$ integer.

The violation of baryonic symmetry generated by quantum anomalies is expected to play an important role for generating the observed baryon asymmetry in our universe [26]. Notice that $B - L$ is however conserved. It is therefore possible to convert a leptonic asymmetry into a baryonic one and vice-versa.

8.3 Gauge anomalies

For *gauge symmetries* on the other hand, quantum anomalies, if present as in fig. (21) generate inconsistencies [27]. Indeed, they would violate *gauge invariance* of the theory, since the gauge variation of the lagrangian from the Noether theorem (5) is

$$\delta\mathcal{L} \sim \alpha_A \, \partial^m J_m^A \ . \tag{393}$$

On the other hand, gauge invariance at the quantum level is crucial for a consistent quantization of the theory, in particular for the decoupling of un-physical states. The corresponding gauge currents are of chiral type

$$J_m^A = \bar{\Psi}_L \gamma_m T_L^A \Psi_L + \bar{\Psi}_R \gamma_m T_R^A \Psi_R \tag{394}$$

and according to our computations above their divergence is proportional to

$$\partial^m J_m^A \;=\; + \;\frac{g_B g_C}{32\pi^2}\, d^{ABC} \epsilon^{mnpq}\, F_{mn}^B\, F_{pq}^C \,. \tag{395}$$

The anomaly coefficients that have to vanish are then

$$2d^{ABC} \;=\; tr\,(\{T^A, T^B\}T^C)_L \;-\; tr\,(\{T^A, T^B\}T^C)_R \;=\; 0\,, \tag{396}$$

where the trace is taken over *all the fermions* in the theory. Notice first that non-chiral (Dirac) fermion matter does not contribute to gauge anomalies. It can also be shown that fermions in *real representations* of the gauge group, for which by definition fermion mass terms are compatible with the gauge invariance, do not contribute to gauge anomalies. On the other hand, fermions in *complex representations of the gauge group*, for which no standard mass terms are possible, do contribute. And this is precisely the case of the quarks and leptons in the Standard Model. By using properties of the Pauli matrices and more generally properties of $SU(2)$ representations, it can checked that there are no pure cubic $SU(2)^3$ anomalies. For the Standard Model, it is actually easy to prove that the only possible gauge anomalies are

$$SU(2)_L^2 U(1)_Y \quad , \quad U(1)_Y^3 \quad \text{and} \quad SU(3)_c^2 U(1)_Y\,. \tag{397}$$

By using the quantum numbers of the known quarks and leptons, the gauge anomaly coefficients in the Standard Model turn out to be

$$\begin{aligned}
&tr\,(\{\frac{\tau^a}{2}, \frac{\tau^b}{2}\}Y)_L \;=\; \frac{1}{2}\delta^{ab}(trY)_L = 3 \times (N_c \times \frac{1}{3} - 1) = 0\,, \\
&tr\,(\{Y, Y\}Y)_{L-R} = \cdots = 6(-2N_c + 6) = 0\,, \\
&tr\,(\{\frac{\lambda^A}{2}, \frac{\lambda^B}{2}\}Y)_{L-R} \;=\; \frac{1}{3}\delta^{AB}(trY)_{L-R} = \cdots = 0\,,
\end{aligned} \tag{398}$$

where in the last eq. in (398) λ^A are the $SU(3)$ Gell-Mann matrices. Notice that anomaly cancelation happens *precisely for three quark colors* $N_c = 3$! This seems to provide a deep *connection between quarks and leptons* in the Standard Model, and a possible *hint towards Grand Unified Theories*. Gauge anomaly cancelation gives generally strong constraints on the possible spectrum of *new chiral particles*. For example, it is easy to show that (homework) :

- the only flavor-independent, anomaly-free Z' with the chiral SM spectrum is $U(1)_{B-L}$, defined according to

Field	q_i	u_{iR}	d_{iR}	l_i	e_{iR}
$\mathrm{U(1)_{B-L}charge}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	-1	-1

- an extension of the Standard Model with a fourth lepton generation l_4, E_R alone (without corresponding quarks), with usual leptonic quantum numbers, is inconsistent with the gauge anomaly cancelation conditions.

Let us finally discuss some subtleties related to the freedom of choosing the shift vectors or, equivalently, of reshuffling the anomaly among the three currents, present in the abelian case. In this case, the three currents are on the same footing, and it is natural to consider a symmetric distribution of anomalies among the three curents. There is a single choice that is fully symmetric, namely

$$a_1 = \frac{1}{3}(k_1 - k_2)\,. \tag{399}$$

We now proceed to construct the effective action for the gauge fields after integrating out the fermions. In what follows we consider for simplicity only abelian gauge fields, but the result can be genralized to the non-abelian case. To cubic order we obtain

$$S_{abc} = \frac{1}{3!} \int d^4x\, d^4y\, d^4z\, \Gamma_{abc}^{\mu\nu\rho}(x, y, z)\, A_\mu^a(x)\, A_\nu^b(y)\, A_\rho^c(z)\,, \tag{400}$$

where no summation is assumed on the a, b, c labels.

Upon gauge transformations $A^a_\mu \to A^a_\mu + \partial_\mu \varepsilon^a$ we obtain

$$\delta S_{abc} = -\frac{1}{3!}\int \left[\varepsilon^a \partial_\mu \Gamma^{\mu\nu\rho}_{abc} A^b_\nu(y)\ A^c_\rho(z) + \varepsilon^b \partial_\nu \Gamma^{\mu\nu\rho}_{abc} A^a_\mu(x)\ A^c_\rho(z) + \varepsilon^c \partial_\rho \Gamma^{\mu\nu\rho}_{abc} A^a_\mu(x)\ A^b_\nu(y)\right] \quad (401)$$

If the shift vector a_1 is a constant independent of momenta then it does not contribute to the gauge variations. We therefore parameterize the scheme dependence as

$$a_1 = Ak_1 + Bk_2 \,. \quad (402)$$

The real numbers A, B can be different for different abc combinations :

$$\begin{aligned}\delta S_{abc} \quad = \quad & -\frac{d_{abc}}{3!(32\pi^2)}\int d^4x\, \Big\{(A-B)_{abc}\varepsilon^a\ \epsilon^{\mu\nu\rho\sigma}\ F^b_{\mu\nu}F^c_{\rho\sigma} + (B_{abc}+1)\varepsilon^b\ \epsilon^{\mu\nu\rho\sigma}\ F^a_{\mu\nu}F^c_{\rho\sigma} \\ & -(A_{abc}-1)\varepsilon^c\ \epsilon^{\mu\nu\rho\sigma}\ F^a_{\mu\nu}F^b_{\rho\sigma}\Big\} \,. \end{aligned} \quad (403)$$

In the case where the currents J_b, J_c are conserved and the whole anomaly is in J_a, then $A_{abc} = -B_{abc} = 1$ and one gets

$$\delta S_{abc} \;=\; -\frac{d_{abc}}{24\pi^2}\int d^4x\ \varepsilon^a\ F^b \wedge F^c \,, \quad (404)$$

where we used

$$F^a \wedge F^b = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\ F^a_{\mu\nu}F^b_{\rho\sigma} \,. \quad (405)$$

In the case where all gauge currents are abelian, it is more natural to use the symmetric scheme $A_{abc} = -B_{abc} = 1/3$. In this case we obtain the gauge variation

$$\delta S_{abc} \;=\; -\frac{d_{abc}}{3!(12\pi^2)}\int d^4x\, \Big\{\varepsilon^a\ \ F^b\wedge F^c + \varepsilon^b\ \ F^a \wedge F^c + \varepsilon^c\ \ F^a\wedge F^b\Big\} \,. \quad (406)$$

In the abelian case, one can sum over the U(1)'s to obtain the full cubic effective action in the symmetric scheme. Its gauge variation is

$$\delta S_3 = \sum_{a,b,c} \delta S_{abc} = -\frac{d_{abc}}{24\pi^2}\int d^4x\ \varepsilon^a F^b \wedge F^c \,, \quad (407)$$

where we have reinstated our summation convention.

If the three currents are abelian, there is clearly an ambiguity in the scheme (choice of the shift vector a_1) in the distribution of the anomaly over the currents. One may study the effect of changing the scheme of the triangle graphs. This is obtained by setting

$$A_{abc} = \frac{1}{3} + \tilde{A}_{abc} \quad , \quad B_{abc} = -\frac{1}{3} + \tilde{B}_{abc} \,. \quad (408)$$

The gauge variation now becomes

$$\begin{aligned}\delta S_3 \quad = \quad & -\frac{d_{abc}}{24\pi^2}\int d^4x\, \Big\{\varepsilon^a\ \ F^b \wedge F^c\Big\} \\ & -\frac{d_{abc}}{3!(8\pi^2)}\int d^4x \Big\{\tilde{A}_{abc}(\varepsilon^a F^b \wedge F^c - \varepsilon^c F^a \wedge F^b) - \tilde{B}_{abc}(\varepsilon^a F^b \wedge F^c - \varepsilon^b F^a \wedge F^c)\Big\}\end{aligned} \quad (409)$$

The extra terms have the same transformation properties as

$$S_{\rm GCS} = -\frac{d_{abc}}{3!(16\pi^2)}\int d^4x\ \epsilon^{\mu\nu\rho\sigma}\left[\tilde{A}_{abc}\ A^a_\mu A^c_\nu F^b_{\rho\sigma} - \tilde{B}_{abc}\ A^a_\mu A^b_\nu F^c_{\rho\sigma}\right] \,. \quad (410)$$

Therefore, in this new scheme, the new effective action is obtained from the old one by adding the so-called generalized Chern-Simon terms (GCS) terms [44] in (410).

A more direct way to see this is to compute the variation of the effective action between two different regularisation schemes specified by the shift vectors a_1^{abc} and $\tilde{a}_1^{abc}$, where $a^{abc} = A_{abc}k_1 + B_{abc}k_2$:

$$\Delta\Gamma^{\mu\nu\rho}_{abc}(x,y,z) = \Gamma^{\mu\nu\rho}_{abc}|_{\tilde{a}_1} - \Gamma^{\mu\nu\rho}_{abc}|_{a_1} \ . \tag{411}$$

By Taylor expanding

$$\begin{aligned}\Delta\Gamma^{\mu\nu\rho}(p,k_i,a_i) &= (\tilde{a}_1 - a_1)^\sigma \frac{\partial}{\partial a_1^\sigma}\Gamma^{\mu\nu\rho}(p,k_i,a_1) \\ &\quad + \frac{1}{2}(\tilde{a}_1 - a_1)^{\sigma_1}(\tilde{a}_1 - a_1)^{\sigma_2}\frac{\partial^2}{\partial a_1^{\sigma_1}\partial a_1^{\sigma_2}}\Gamma^{\mu\nu\rho}(k_i,a_1) + \cdots \end{aligned}\tag{412}$$

and noticing that $\partial\Gamma^{\mu\nu\rho}(p,k_i,a)/\partial a^\sigma = \partial\Gamma^{\mu\nu\rho}(p,k_i,a)/\partial p^\sigma$, we can cast the scheme difference into the form

$$\begin{aligned}\Delta\Gamma^{\mu\nu\rho}_{abc}(x,y,z) &= \frac{i}{(2\pi)^{12}}\int d^4k_1 d^4k_2\, e^{-i(k_1+k_2)x+ik_1y+ik_2z}(\tilde{a}_1 - a_1)^\sigma \times \\ &\quad \int d^4p\, t_{abc}\frac{\partial}{\partial p^\sigma}\left[\Gamma^{\nu\rho\mu}_{abc}(p,k_1,k_2,a_1) - \Gamma^{\rho\nu\mu}_{abc}(p,k_2,k_1,-a_1)\right] + \cdots \ , \end{aligned}\tag{413}$$

where $\cdots$ are contributions at least quadratic in the shift vectors a containing at least second derivatives with respect to the loop momentum p. Since all contributions come from the boundary of the loop momentum space, we will see in a moment that only the first contribution gives a non-vanishing contribution. Like in the case of the triangle gauge anomalies, the quantity $\Delta\Gamma^{\mu\nu\rho}_{abc}$ is given by a surface contribution. A simple counting of the leading momentum dependence for $p\to\infty$ shows that only the leading contribution

$$\Gamma^{\nu\rho\mu}_{ijk}(p,k_1,k_2,a_1) \to -\frac{2}{p^6}\left[p^2(p^\mu\eta^{\nu\rho} + p^\nu\eta^{\mu\rho} + p^\rho\eta^{\mu\nu}) - 4p^\mu p^\nu p^\rho + ip^2\epsilon^{\nu\rho\mu\sigma}p_\sigma\right] \tag{414}$$

is giving a non-vanishing result and only the last term in (414) contributes to (413). By explicitly computing now the surface integral

$$\int d^4p\, \partial_\sigma \frac{p_\epsilon}{p^4} = -\frac{1}{8}\eta_{\sigma\epsilon}\int d^4p\, \partial^2\frac{1}{p^2} = -\frac{\pi^2}{4}\eta_{\sigma\epsilon} \ , \tag{415}$$

we finally get the difference of the effective action in two different regularisation schemes to be equal to

$$\begin{aligned}\Delta S_3^{\rm an} &= \frac{1}{3!}\int d^4x\, d^4y\, d^4z\, \Delta\Gamma^{\mu\nu\rho}_{abc}(x,y,z)\, A^a_\mu(x)\, A^b_\nu(y)\, A^c_\rho(z) \\ &= \frac{1}{32\pi^2}d_{abc}\, (\Delta A_{acb} - \Delta B_{abc})\int A^a\wedge A^b\wedge F^c \ . \end{aligned}\tag{416}$$

9 Effective action and the effective potential

The starting point is the addition of an external source $J(x)$ to the lagrangian density

$$\mathcal{L}(\phi,\partial_m\phi) \ \to \ \mathcal{L}(\phi,\partial_m\phi) \ + \ J(x)\phi(x) \ . \tag{417}$$

The generating functional $W(J)$ of Green functions is

$$e^{iW(J)} \ = \ \langle 0,{\rm out}|0,{\rm in}\rangle_J = \ \langle 0,{\rm out}|\ e^{i\int d^4x J(x)\phi(x}\ |0,{\rm in}\rangle. \tag{418}$$

The expansion of $W(J)$ in a Taylor functional series generates the connected Green functions $G^{(n)}(x_1 \cdots x_n)$

$$W(J) \;=\; \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \; G^{(n)}(x_1 \cdots x_n) \; J(x_1) \cdots J(x_n) \; . \tag{419}$$

The classical field ϕ_c is defined by

$$\phi_c(x) \;=\; \frac{\delta W}{\delta J(x)} \;=\; \frac{\langle 0, \mathrm{out} | \phi(x) | 0, \mathrm{in} | \rangle_J}{\langle 0, \mathrm{out} | 0, \mathrm{in} | \rangle_J} . \tag{420}$$

The effective action $\Gamma(\phi_c)$ is defined by the Lagrange transform

$$\Gamma(\phi_c) \;=\; W(J) \;-\; \int d^4x \; J(x) \phi_c(x) \; , \tag{421}$$

from which it follows that

$$\frac{\delta \Gamma}{\delta \phi_c(x)} \;=\; - \; J(x) \; . \tag{422}$$

An illuminating intuitive interpretation of $\Gamma(\phi_c)$ can be found using the path-integral formalism for field theory, according to which

$$Z(J) = e^{iW(J)} = \int \mathcal{D}\phi \; e^{i \int d^4x [\mathcal{L}(\phi) + J(x)\phi(x)]} \; . \tag{423}$$

The definition of the effective action (421) used in (423) leads then to

$$e^{i\{\Gamma(\phi_c) + \int d^4x \; J(x)\phi_c(x)\}} = \int \mathcal{D}\phi \; e^{i \int d^4x [\mathcal{L}(\phi) + J(x)\phi(x)} \; . \tag{424}$$

Therefore the effective action is that of a *classical field* $\phi_c(x)$ that reproduces the spectrum and interactions of a *quantum field* $\phi(x)$ of action $S[\phi(x)] = \int d^4x \mathcal{L}(\phi)$, in the presence of an external source J. When the source is switched-off, the effective action correctly captures the extrema of the quantum system.

Analogously to (419), the expansion of $\Gamma(\phi_c)$ in a Taylor functional series generates the one-particle irreducible (1PI) Green functions $\Gamma^{(n)}(x_1 \cdots x_n)$

$$\Gamma(\phi_c) \;=\; \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \; \Gamma^{(n)}(x_1 \cdots x_n) \phi_c(x_1) \cdots \phi_c(x_n) \; . \tag{425}$$

By definition, the 1PI Green functions are Green functions which cannot be separated into two disconnected diagrams by cutting one internal line/propagator. In addition, in the corresponding Feynman rules, the external propagators should be removed. More precisely, we have

$$\Gamma^{(n)}(x_1 \cdots x_n) \;=\; -i \; \langle \phi(x_1) \cdots \phi(x_n) \rangle_{\mathrm{1PI}} \; . \tag{426}$$

The effective action has an alternative expansion in derivatives

$$\Gamma(\phi_c) \;=\; \int d^4x \left[-V(\phi_c) \;+\; \frac{1}{2} Z(\phi_c)(\partial\phi_c)^2 + \cdots \right] \; , \tag{427}$$

where $\cdots$ denote higher derivative terms. In (427), $V(\phi_c)$ is called the effective potential. In order to express it as a function of the irreducible Green functions, let us Fourier tranform

$$\Gamma^{(n)}(x_1 \cdots x_n) = \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \; (2\pi)^4 \; \delta^4(k_1 + \cdots k_n) \; e^{i(k_1 x_1 + \cdots k_n x_n)} \; \Gamma^{(n)}(k_1 \cdots k_n) \; . \tag{428}$$

By using (425), (427) and (428), one finds

$$V(\phi_c) \;=\; -\sum_n \frac{1}{n!}\Gamma^{(n)}(0\cdots 0)\,\phi_c^n\;. \tag{429}$$

Its classical limit is the usual scalar potential, but for finite $\hbar$, $V(\phi_c)$ contains the quantum corrections to it. As such, it plays an important role since it encodes the quantum corrections to the ground state and the quantum aspects of spontaneous symmetry breaking. Indeed, from (422) one sees that by switching off the source J we reach an extremum of the effective action, which for translationally invariant vacua is equivalent to an extremum of the effective potential

$$\frac{\delta\Gamma}{\delta\phi_c} = 0 \quad \to \quad \frac{dV}{d\phi_c} = 0\;. \tag{430}$$

The computation of the effective action at the loop level will be plagued with UR (and possibly IR) divergences. We will therefore impose renormalisation conditions for masses, couplings and wavefunctions. In what follows we will use the counterterms method and a simple momentum cutoff regularization .

Let us first of all show that the quantum expansion of the effective action in powers of $\hbar$ is equivalent to the expansion in the number of loops. In the functional integral quantization, the lagrangian appears as

$$\frac{1}{\hbar}\mathcal{L}(\phi,\partial\phi)\;. \tag{431}$$

Let us call P the power of $\hbar$ associated to any graph, than can be expressed as a function of the internal propagators I and the number of vertices V of the graph as

$$P = I - V\;, \tag{432}$$

since every propagator carries a factor of $\hbar$, while every vertex carries a factor of $\hbar^{-1}$. On the other hand, the number of loops is given by

$$L = I - V + 1\;, \tag{433}$$

where the number of loops can be conveniently defined as the number of independent integration momenta. Combining (432) and (433), one finds indeed

$$P = L - 1\;, \tag{434}$$

which proves the equivalence of the semiclassical expansion and the loop expansion.

The computation of the effective potential can be done diagramatically or by functional methods. In the diagrammatic approach, one starts from (425), for constant classical fields ϕ_c. Let us consider for definiteness a massless scalar theory with quartic self-interaction, of lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}A(\partial\phi)^2 - \frac{1}{2}B\phi^2 - \frac{1}{4!}C\phi^4\;, \tag{435}$$

where the last three terms in (435) are counterterms that will be fixed by renormalization conditions. The simplest, naive renormalization conditions we would like to impose in the effective potential $V(\phi_c)$ for such a massless theory are

$$\frac{d^2V}{d\phi_c^2}|_{\phi_c=0} = 0 \quad , \quad \frac{d^4V}{d\phi_c^4}|_{\phi_c=0} = \lambda\;. \tag{436}$$

To lowest-order in perturbation theory, only one graph contributes and

$$V_{tree} = \frac{\lambda}{4!}\phi_c^4\;. \tag{437}$$

At the one-loop level there are an infinity of polygon Feynman diagrams with $2n$ external lines of scalar fields ϕ_c, computed at zero external momenta. By adding to them the tree-level contribution (437) and the counterterms, one finds

$$V(\phi_c) = V_{tree} + V_{1-loop} = \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}B\phi_c^2 + \frac{1}{4!}C\phi_c^4$$
$$+i\int \frac{d^4p}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\lambda\phi_c^2/2}{p^2+i\epsilon}\right)^n \ . \tag{438}$$

The numerical factors in (438) have the following explanation:
- the overall factor of i comes from the definition of $W(J)$, eq. (418).
- The factor of $1/2$ in $\lambda\phi_c^2/2$ comes from the incomplete cancelation of the $1/4!$ in the interaction, since the interchange of the two external lines at the same vertes does not lead to a new graph.
- The factor of $1/2n$ has a combinatorial origin: rotation or reflection of the polygon with $2n$ external scalar lines does not lead to an independent contraction in the Wick expansion. This leads to an incomplete cancelation of the $1/n!$ factor in the perturbation theory formula.

Notice that each term in the one-loop series is severely IR divergent, due to the n massless scalar propagators. A much better situation is obtained by realizing that one can sum the series. After the Wick rotation to the Euclidian space, one obtains

$$V(\phi_c) = \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}B\phi_c^2 + \frac{1}{4!}C\phi_c^4 + \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\ln\left(1+\frac{\lambda\phi_c^2}{2p^2}\right) \ . \tag{439}$$

One can see that the resulting potential has now only a logarithmic singularity at the origin $\phi_c = 0$. The explicit computation with a momentum cutoff Λ gives the result

$$V(\phi_c) = \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}B\phi_c^2 + \frac{1}{4!}C\phi_c^4 + \frac{1}{32\pi^2}\left[\frac{\lambda\phi_c^2\Lambda^2}{2} + \frac{\lambda^2\phi_c^2}{8}\left(\ln\frac{\lambda\phi_c^2}{2\Lambda^2} - \frac{1}{2}\right)\right] \ . \tag{440}$$

One novelty is that, due to the singularity at the origin, it is not possible to use the renormalization conditions (436). Instead, one is forced to introduce an arbitrary mass scale M into the theory and to modify the renormalization conditions according to

$$\frac{d^2V}{d\phi_c^2}|_{\phi_c=0} = 0 \quad , \quad \frac{d^4V}{d\phi_c^4}|_{\phi_c=M} = \lambda \ . \tag{441}$$

The new renormalization conditions (441) determine the counterterms

$$B = -\frac{\lambda\Lambda^2}{32\pi^2} \quad , \quad C = -\frac{3\lambda^2}{32\pi^2}\left(\ln\frac{\lambda M^2}{2\Lambda^2} + \frac{11}{3}\right) \ . \tag{442}$$

The final renormalized one-loop corrected effective potential becomes then

$$V(\phi_c) = \frac{\lambda}{4!}\phi_c^4 + \frac{\lambda^2\phi_c^4}{256\pi^2}\left(\ln\frac{\phi_c^2}{M^2} - \frac{25}{6}\right) \ . \tag{443}$$

As expected for a renormalizable theory, the dependence on the cutoff disappeared in the final renormalized expression (443). As emphasized before, the renormalized mass scale M is arbitrary; the physics should be independent on it. One can equally well use a different value M' in the renormalization conditions (441). In this case, one obtains the effective potential

$$V(\phi_c) = \frac{\lambda'}{4!}\phi_c^4 + \frac{\lambda'^2\phi_c^4}{256\pi^2}\left(\ln\frac{\phi_c^2}{M'^2} - \frac{25}{6}\right) \ . \tag{444}$$

Since the effective potential is a physical quantity, it should be the same for any value used for M. We obtain therefore a relation between the couplings λ and λ'

$$\lambda - \frac{3\lambda^2}{32\pi^2}\ln M^2 = \lambda' - \frac{3\lambda'^2}{32\pi^2}\ln M'^2 \ , \tag{445}$$

that at the one-loop level can be simplified to

$$\lambda' = \lambda + \frac{3\lambda^2}{32\pi^2}\ln\frac{M'^2}{M^2} \ . \tag{446}$$

It is gratifying that by using the effective potential we obtain the same equation (309) obtained by diagrammatic methods, leading to the RG equation (325). The effective potential can therefore be used to derive in a compact way RG equations for couplings of the theory.

At the technical level, the computation of the effective potential is simpler in the path integral formulation. In this approach, one expands the quantum field around a classical value

$$\phi = \phi_c + \phi_q \ , \tag{447}$$

where ϕ_q describe oscillations of the quantum field around the classical configuration ϕ_c. One-loop is equivalent to expanding to the quadratic order in the fluctuation. One then expands

$$\begin{aligned} S(\phi) + \int d^4x J(x)\phi(x) &= S(\phi_c) + \int d^4x J(x)[\phi_c(x) + \phi_q(x)] \\ + \int d^4x \phi_q(x)\left[\frac{\delta S}{\delta\phi(x)}\right]_{\phi_c} &+ \frac{1}{2!}\int d^4x_1 \int d^4x_2 \phi_q(x_1)\left[\frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)}\right]_{\phi_c}\phi_q(x_2) + \cdots \end{aligned} \tag{448}$$

The linear term in ϕ_q in (448) vanishes because of (422). Substituting (448) into (426) and performing the gaussian integral, one finds

$$e^{i\{\Gamma(\phi_c)+\int d^4x \ J(x)\phi_c(x)\}} = e^{i\int d^4x \ [\mathcal{L}+J(x)\phi_c(x)]} \times \left[Det\frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)}\right]^{-\frac{1}{2}} \ , \tag{449}$$

or the equivalent relation

$$\Gamma(\phi_c) \ = \ S(\phi_c) + \frac{i}{2}tr\ln\left[\frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)}\right] \ , \tag{450}$$

where in (449) and (450) the determinant and the trace are defined in the functional sense. For the massless ϕ^4 theory that we discussed in this section,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4!}\phi^4 \quad \to \quad \frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)} = -(\Box + \frac{\lambda\phi_c^2}{2})\delta^4(x_1 - x_2) \tag{451}$$

and after passing to the momentum space and the Wick rotation, one is led after adding the counterterms the same effective potential (439) that we found by the diagrammatic method, up to a field-independent contribution to the vacuum energy.

Notice that at the one-loop level one can generalize the above considerations to a general formula. Let us consider a theory containing scalar or fermionic fields $\chi, \bar{\chi}$ and gauge fields A_m, coupled to the scalar ϕ for which we want to compute the effective action. Then, at the one-loop order,

$$e^{i\Gamma(\phi_c)} \ = \ \int \mathcal{D}(\phi_q, \chi, \bar{\chi}, A_m) \ e^{iS_{\rm quadratic}(\phi_c+\phi_q,\chi,\bar{\chi},A_m)} \ , \tag{452}$$

where $S_{\rm quadratic}(\phi_c + \phi_q, \chi, \bar{\chi}, A_m)$ denotes the classical action containing the classical contribution plus the quadratic terms in the quantum fields. One can writes an explicit form

$$S_{\rm quadratic}(\phi_c + \phi_q, \chi, \bar{\chi}, A_m) = S(\phi_c) + \frac{1}{2}\phi_q M_\phi^2(\phi_c)\phi_q + \chi M_\chi^2(\phi_c)\bar{\chi} + \frac{1}{2}A_m M_A^2(\phi_c)A^m \ , \quad (453)$$

where $M_\phi^2(\phi_c)$,etc are *field-dependent* masses, which reduce to the true mass in the ground state $\langle\phi\rangle$. For constant fields ϕ_c one can then find a compact form for the effective potential [11]

$$V(\phi_c) = V_{tree} + \frac{1}{2} Str \int \frac{d^4k}{(2\pi)^4} \ln(k^2 + M^2(\phi_c)) \ , \quad (454)$$

where $Str M^2(\phi_c) = Tr(M_s^2)(\phi_c) - 2Tr(M_f^\dagger M_f)(\phi_c) + 3Tr(M_v^2)(\phi_c)$ and where $M_s(\phi_c)$, $M_f(\phi_c)$ and $M_v(\phi_c)$ denote (real) scalar, (Weyl) fermionic and spin-1 vector field-dependent mass matrices, respectively.

9.1 Application: Running of λ in the Standard Model from top loops

The top Yukawa coupling is the only one which is large, very close to one. As such, it has the largest effect on radiative corrections and in particular in the energy evolution of the other couplings in the Standard Model. Let us use the effective potential formalism in order to work out the effect of the top Yukawa on the running of the Higgs self-coupling λ. The relevant effective lagrangian in this case is

$$\mathcal{L}_{\rm Higgs-top} = \mathcal{L}_{\rm kin.} - \lambda(\Phi^\dagger\Phi)^2 - (h_t \bar{t}_L t_R \tilde{\Phi} + {\rm h.c.}) + \cdots \ , \quad (455)$$

The Higgs (field)-dependent top mass, to be used in the effective potential computation is

$$(M_F^\dagger M_F)(\Phi_c) = h_t^2 \Phi_c^\dagger \Phi_c \ . \quad (456)$$

According to (454), the top loops induce a one-loop contribution

$$V_{1-loop} = -3 \times 4 \times \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln(k^2 + h_t^2 \Phi_c^\dagger \Phi_c) \ , \quad (457)$$

where the factor of 3 in front counts the number of colors and 4 is the contribution of a Dirac fermion. Evaluating explicitly with a UV momentum cutoff, one finds

$$V_{1-loop} = -\frac{3h_t^2 \Phi_c^\dagger \Phi_c \Lambda^2}{8\pi^2} - \frac{3h_t^4}{16\pi^2}(\Phi_c^\dagger\Phi_c)^2 \left(\ln \frac{h_t^2 \Phi_c^\dagger \Phi_c}{\Lambda^2} - \frac{1}{2}\right) \ . \quad (458)$$

Adding the tree-level potential and the counterterms, the full one-loop effective potential becomes

$$\begin{aligned} V_(\Phi_c) &= -\mu^2|\Phi_c|^2 + \lambda|\Phi_c|^4 + B|\Phi_c|^2 + C|\Phi_c|^4 \\ &- \frac{3h_t^2|\Phi_c|^2\Lambda^2}{8\pi^2} - \frac{3h_t^4}{16\pi^2}|\Phi_c|^4 \left(\ln \frac{h_t^2|\Phi_c|^2}{\Lambda^2} - \frac{1}{2}\right) \ . \end{aligned} \quad (459)$$

By imposing the renormalization conditions

$$[\frac{\partial^2 V}{\partial\Phi\partial\Phi^\dagger}]_{\Phi_c=0} = -\mu^2 \ , \ [\frac{\partial^4 V}{\partial\Phi^2\partial\Phi^{\dagger 2}}]_{\Phi_c=M} = 4\lambda \ , \quad (460)$$

one finds the renormalized effective potential

$$V_(\Phi_c) = -\mu^2|\Phi_c|^2 + \lambda|\Phi_c|^4 - \frac{3h_t^4}{16\pi^2}|\Phi_c|^4 \left(\ln \frac{|\Phi_c|^2}{M^2} - 3\right) \ . \quad (461)$$

[11] We ignore here the counterterms, that have to be added of course for appropriate renormalization.

In analogy with (445) and (446), one can find the running of λ by imposing the invariance of the effective potential under changes of the renormalization scale M. One finds

$$\lambda(M) + \frac{3h_t^4}{16\pi^2} \ln M^2 = \lambda(M') + \frac{3h_t^4}{16\pi^2} \ln M'^2 \,, \tag{462}$$

or equivalently

$$\lambda(M') = \lambda(M) - \frac{3h_t^4}{16\pi^2} \ln \frac{M'^2}{M^2} \,. \tag{463}$$

The effect of the top loops is therefore a decrease of the self-coupling λ by going towards higher energy. There is the danger that the coupling becomes negative at some scale. We will analyze this phenomenon in more detail and its interpretation of in the next section.

9.2 Spontaneous symmetry breaking by radiative corrections ?

Starting from the original massless theory with no symmetry breaking pattern, one can show that in the quantum corrected one-loop theory, a non-trivial minimum is generated leading to spontaneous symmetry breaking [45]. Indeed, minimization of the one-loop effective potential (443) leads to an extremum for $\phi_c = \langle\phi\rangle$ determined by

$$\lambda \ln \frac{\langle\phi\rangle^2}{M^2} = -\frac{32\pi^2}{3} + \frac{11\lambda}{3} \,. \tag{464}$$

The fact that this is a minimum can be seen be computing

$$\begin{aligned} &\frac{d^2V}{d\phi_c^2}|_{\langle\phi\rangle} = \frac{\lambda\langle\phi\rangle^2}{2} + \frac{3\lambda^2\langle\phi\rangle^2}{64\pi^2}\left(\ln\frac{\langle\phi\rangle^2}{M^2} - 3\right) \simeq \frac{\lambda\langle\phi\rangle^2}{2} \ > \ 0 \,, \\ &\langle V\rangle = -\frac{\lambda^2\langle\phi\rangle^4}{512\pi^2} \ < \ 0 \,. \end{aligned} \tag{465}$$

The quantum generated minimum is therefore the absolute ground state of the theory. However, by taking into account possible higher-order corrections, one realizes that this minimum lies far outside the validity of the perturbation theory and therefore it is not reliable. As shown in [45] however, in other interesting theories like for example scalar electrodynamics, such a loop induced spontaneous symmetry breaking is realized in perturbation theory and lead to the interesting phenomenon of *dimensional transmutation*. This means that a dimensionless coupling of the initial massless theory is traded for a dimensional mass parameter M in the quantum theory, that will trigger spontaneous symmetry breaking and set the scale for physical masses.

Notice that the original ϕ^4 self-interacting theory has the classical scale invariance

$$x'_m \ = \ e^{-\zeta}\, x_m \quad , \quad \phi' \ = \ e^{\zeta}\, \phi \,, \tag{466}$$

with ζ a constant parameter. In the whole Standard Model lagrangian, the only dimensionfull parameter breaking the scale invariance (466) and acting also on gauge fields A_m and fermions Ψ as

$$\Psi' \ = \ e^{\frac{3\zeta}{2}}\, \Psi \quad , \quad A'_m \ = \ e^{\zeta}\, A_m \,, \tag{467}$$

is the Higgs mass parameter μ^2. This mass scale was the responsible for the spontaneous symmetry breaking $SU(2)_L \times U(1)_Y \to U(1)_{\rm em}$ and the generation of the electroweak scale. But the value of the electroweak scale $v \simeq 246$ GeV raises the important question of its origin (see also the next Section). The fact that spontaneous symmetry breaking can also occur in a classically scale-invariant theory opens the interesting possibility of the generation of the weak scale itself, in analogy with the QCD scale, by dimensional transmutation.

10 The Higgs / Symmetry breaking sector of the Standard Model.

The Higgs boson is the last building block of the Standard Model awaiting its experimental discovery. There are good theoretical reasons to and preliminary experimental hints from LHC to hope that this can happen quite soon. We review here some of the theoretical biases which make theorists to favor the existence of a light higgs scalar. The first two, the perturbativity and the stability bound, are obtained by extrapolating the Standard Model to high energy scales and imposing perturbativity of couplings and stability of the SM ground state, respectively. The third one is related to the breakdown of unitarity in the longitudinal WW scattering at high-energy if the higgs is too heavy or if does not exist.

10.1 Perturbativity bounds

As shown in Section 7, in quantum field theory couplings run. The Higgs mass is then obtained by knowing the Higgs self-coupling λ at the electroweak scale $M_h^2 = 2\lambda(v)v^2$. If this coupling is large enough, it will hit a Landau pole at a high-energy scale called Λ in what follows. The RGE for the *Higgs self-coupling* in the SM is

$$16\pi^2 \frac{d\lambda}{d\ln\mu} = 24\lambda^2 - (3g'^2 + 3g^2 - 12h_t^2)\,\lambda\, + \frac{3}{8}(g'^4 + 2g^2g'^2 + 3g^4) - 6h_t^4\, +\, \cdots\ , \tag{468}$$

where $\cdots$ denote smaller Yukawas. In the *large Higgs mass limit* $\lambda >> g^2, h_t^2$, this reduces to

$$\frac{d\lambda}{\lambda^2} = \frac{3}{2\pi^2}\, d\ln\mu\ \Rightarrow\ \frac{1}{\lambda(\mu)} = \frac{1}{\lambda(\Lambda)} + \frac{3}{2\pi^2}\ln\frac{\Lambda}{\mu}\ . \tag{469}$$

This can be interpreted in two alternative ways [46] :

i) If the *Higgs mass is known*, SM has a Landau pole (signal of a non-perturbative regime) $\lambda(\Lambda) >> 1$ at an energy scale

$$\Lambda\ =\ v\, e^{\frac{2\pi^2}{3\lambda}}\ =\ v\, e^{\frac{4\pi^2 v^2}{3M_h^2}}\ . \tag{470}$$

ii) Conversely, asking for *perturbativity* up to scale Λ (say M_{GUT}), we obtain an *upper bound* on the Higgs mass (homework)

$$M_h^2\ \leq\ \frac{4\pi^2 v^2}{3\ln\frac{\Lambda}{v}}\ . \tag{471}$$

10.2 Stability bounds

Standard Model has a potential instability in the *small Higgs mass limit* [47], since if too small at the electroweak scale, λ can *become negative* at high-energy by the RG running. If $\lambda << h_t^2$, the relevant leading RGE's are

$$16\pi^2 \frac{d\lambda}{d\ln\mu}\ =\ -6h_t^4\ ,\ 16\pi^2 \frac{dh_t}{d\ln\mu}\ =\ \frac{9h_t^3}{2}\ , \tag{472}$$

which integrate to (homework)

$$\begin{aligned} \lambda(\mu)\ &=\ \lambda(\Lambda) + \frac{\frac{3h_t^4(\Lambda)}{8\pi^2}\ln\frac{\Lambda}{\mu}}{1 + \frac{9h_t^2(\Lambda)}{16\pi^2}\ln\frac{\Lambda}{\mu}}\ , \\ h_t^2(\mu)\ &=\ \frac{h_t^2(\Lambda)}{1 + \frac{9h_t^2(\Lambda)}{16\pi^2}\ln\frac{\Lambda}{\mu}}\ . \end{aligned} \tag{473}$$

At the one-loop level, the running of λ induced by the top loos is the same derived explicitly by the effective potential method in (463). As in the perturbativity case limit, this can be interpreted in two ways :

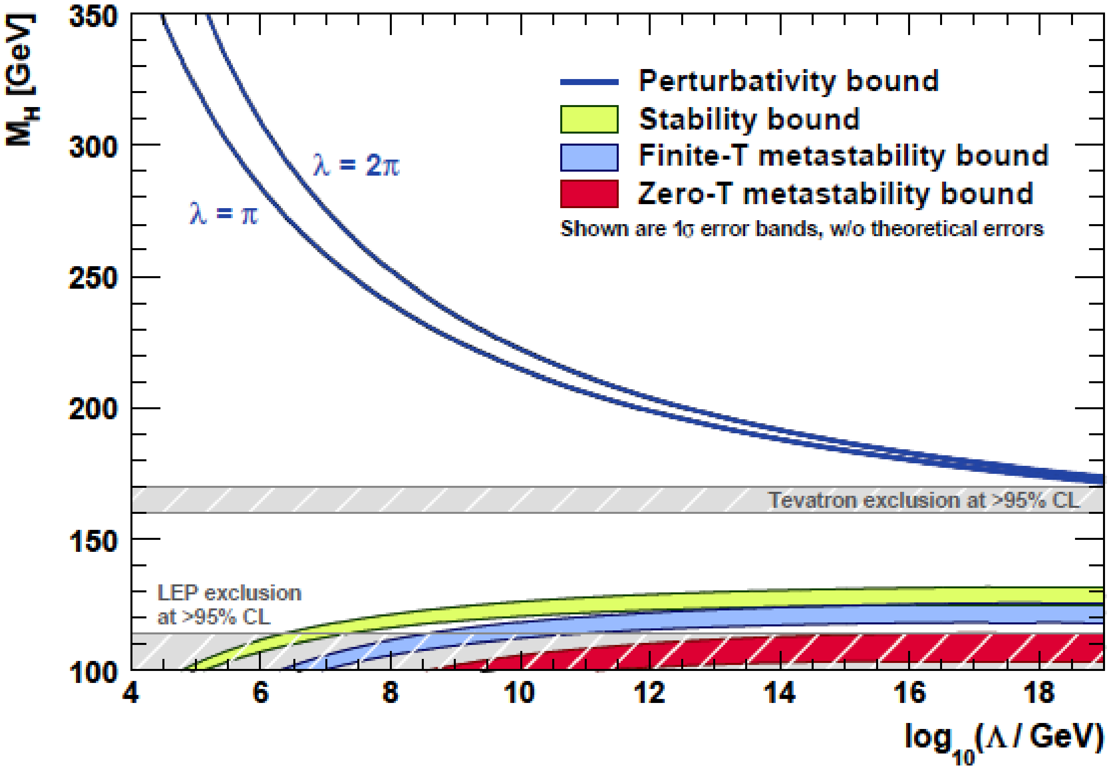


Fig. 22: Perturbativity and stability Higgs mass limits. Λ is the scale of new physics (from [48]).

i) For a fixed, *known value of the Higgs mass*. Let us take $\mu = v$. Then, new physics should show up before the scale Λ where $\lambda(\Lambda) = 0$,

$$\Lambda \; \leq \; v \, e^{\frac{8\pi^2 \lambda}{3h_t^4}} \; = \; v \, e^{\frac{4\pi^2 M_h^2}{3h_t^4 v^2}} \; . \tag{474}$$

ii) Alternatively, for a *fixed* Λ, we get a *lower bound* on the Higgs mass (homework)

$$M_h^2 \; \geq \; \frac{3h_t^4 v^2}{4\pi^2} \ln \frac{\Lambda}{v} \; = \; \frac{3m_t^4}{\pi^2 v^2} \ln \frac{\Lambda}{v} \; . \tag{475}$$

These theoretical *Higgs mass limits* are summarized in the plot in Figure 22, which contain more accurate numerical solution to the RG equations. If the scale Λ is very low, these bounds are very loose. On the other hand, if the SM as an effective theory is valid up to the Planck scale, we obtain a pretty tight mass range $120 \lesssim M_h \lesssim 170$ GeV. Recent discovery of the new boson at LHC of mass of order 126 GeV, most likely the Higgs scalar responsible for the electroweak symmetry breaking, is pointing into the metastability region. In this case, there is another vacuum deeper in energy and therefore the real ground state, at very large vev's of the scalar, whereas the electroweak vacuum is only locally stable. However, the lifetime of our vacuum is estimated to be way beyond the current age of the Universe. It remains to be seen if this metastability can have observable consequences.

10.3 $W\ W$ scattering and unitarity

There is another bound on the higgs mass which does not involve extrapolations of the SM model to very high energies. It is coming from the unitarity of scattering amplitude for the longitudinal $W_L W_L \to W_L W_L$ scattering [49].

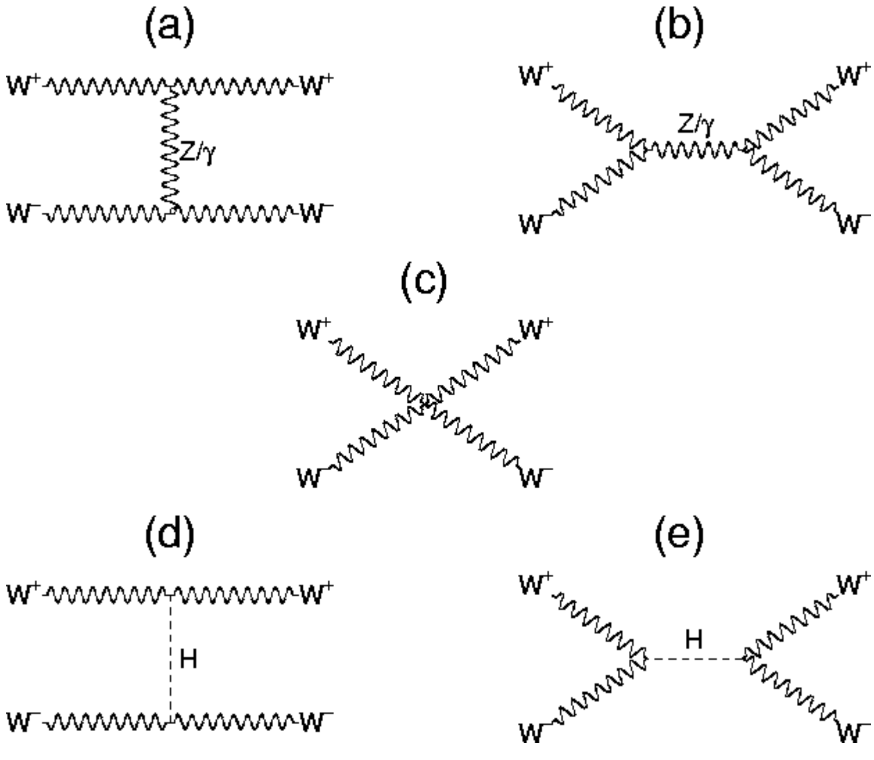


Fig. 23: Tree-level diagrams contributing to WW scattering.

For a massive W gauge particle of momentum k and mass M_W, $A_m = \epsilon_m\, e^{ikx}$, the three polarizations satisfy $\epsilon_m \epsilon^m = -1$, $k_m \epsilon^m = 0$. In the rest frame $k^m = (E, 0, 0, k)$, they are

$$\begin{aligned} &\text{transverse} \ : \ \epsilon_1^m = (0,1,0,0) \quad , \quad \epsilon_2^m = (0,0,1,0) \ , \\ &\text{longitudinal} \ : \ \epsilon_L^m = (\frac{k}{M_W}, 0, 0, \frac{E}{M_W}) \sim \frac{k^m}{M} + \mathcal{O}(\frac{M_W}{E}) \ , \end{aligned} \tag{476}$$

the last expressions being valid for $k \to \infty$. Since longitudinal polarization is proportional to the energy, tree-level amplitude behaves as

$$\mathcal{A} \ = \ \mathcal{A}^{(4)} (\frac{E}{M_W})^4 \ + \ \mathcal{A}^{(2)} (\frac{E}{M_W})^2 + \cdots \ . \tag{477}$$

Actually, the diagrams a),b) and c) in Figure 27 give $\mathcal{A} = g^2 (\frac{E}{M_W})^2$. On the other hand, *unitarity constrains the amplitude* to stay *small enough* at any energy. In order to see this, let us consider the unitarity of the S-matrix $S^\dagger S = 1$. Then

$$S = 1 + i\mathcal{A} \quad \Rightarrow \quad i(\mathcal{A} - \mathcal{A}^\dagger) + \mathcal{A}^\dagger \mathcal{A} = 0 \tag{478}$$

By sandwiching this eq. between a two-particle state $|i>$:

$$i(\mathcal{A} - \mathcal{A}^\dagger)_{ii} + \sum_f |\mathcal{A}_{fi}|^2 = 0 \ , \tag{479}$$

we find the **optical theorem** : *the imaginary part of the forward amplitude of the process $i \to i$ is proportional to the total cross section of* $i \to$ anything.
Let us now decompose the scattering amplitude into partial waves

$$\mathcal{A} \ = \ \sum_{l=0}^{\infty} (2l+1)\ P_l(\cos\theta) a_l \ , \tag{480}$$

where a_l are partial wave amplitudes of the elastic scattering of two particles. Projecting (479) into the partial wave l gives $Im\ a_l = |a_l|^2$. This is only possible if

$$|Re\ a_l| \leq 1/2\ ,\ 0 \leq Im\ a_l \leq 1 \quad \to \quad |a_l|^2 \leq 5/4\ , \tag{481}$$

which is the *unitarity bound* we were searching for.
In the case of the SM without the Higgs boson, diagrams a),b) and c) in Fig. 27 lead to

$$a_0 = \frac{g^2 E^2}{M_W^2} \quad \Rightarrow \quad \text{unitarity breaks down for} \sqrt{s} \sim 1.2\, TeV\ . \tag{482}$$

With the Higgs boson present, amplitudes d),e) in Fig. 27 cancel the raising energy term, such that

$$a_0 = \frac{g^2 M_H^2}{4 M_W^2} \quad \to \quad \text{unitarity breaks down unless } M_H \leq 1.2 \text{ TeV}\ . \tag{483}$$

By considering other channels, one get the stronger bound $M_H \leq 800$ GeV.
Intepretation: If LHC finds no Higgs with a mass $M_H \leq 800 GeV$, unitarity of S-matrix will be violated. New light degrees of freedom should exist in order to restore unitarity. Most theorists interpret this result as a *no-loose "theorem"* for LHC: either LHC finds the Higgs, or it should find the degrees of freedom replacing it in order to unitarize the WW scattering.

It is important to keep in mind however that most BSM models have *invisible higgs decays*. For example, dark matter models can have higgs decays into dark matter particles $h \to\ DM\ DM$. In this case, higss searches are more complicated : the higgs can be "hidden" due to its non-standard decays.

There are other constraints on the Higgs mass that we not discuss here, coming from precision tests in the Standard Model (see Fig. 24). Most theories have a biased towards a *light Higgs*, since it provides a better fit for the SM precision tests.

10.4 Higgs and the hierarchy problem

Quantum corrections to the Higgs mass in the SM, coming from diagrams in Fig. 25, are quadratically divergent

$$\delta m_h^2 \ \simeq \ \frac{3\Lambda^2}{8\pi^2 v^2} (4 m_t^2 - 4 M_W^2 - 2 M_Z^2 - m_h^2)\ . \tag{484}$$

In a theory including gravity or GUT's, Λ is a physical mass scale $\Lambda = M_P, M_{GUT}$. It is then difficult to understand why

$$m_h^2 = (m_h^0)^2 + \frac{3\Lambda^2}{8\pi^2 v^2} (4 m_t^2 - 4 M_W^2 - 2 M_Z^2 - m_h^2) \sim v^2 \ << \ \Lambda^2 \tag{485}$$

This is the *the hierarchy problem* [51].

The latest news before this School, from "Lepton-Photon" in august 2011 concerning the Higgs were that both ATLAS and CMS did exclude the SM Higgs at 95 CL for $145 \ \leq M_H \ \leq 446\ GeV$ except $288 - 296$ GeV. Before the Christmass 2012 however, some excess in the data, first at ATLAS and then at CMS, has been interpreted as the first possible evidence for a Higgs boson around 125 GeV. The figure 26 summarizes the situation in the Moriond 2012 conference [54].

11 Neutrino masses and mixings

11.1 Dirac and Majorana masses. Seesaw mechanism

As we already discussed in Subsection 3.7, there are two possible types of fermion masses. The first one is the Dirac mass, which mixes the two chiralities

$$\mathcal{L}_{\text{Dirac mass}} = -m_D \bar{\Psi}\Psi = -m_D \left(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L\right)\ . \tag{486}$$

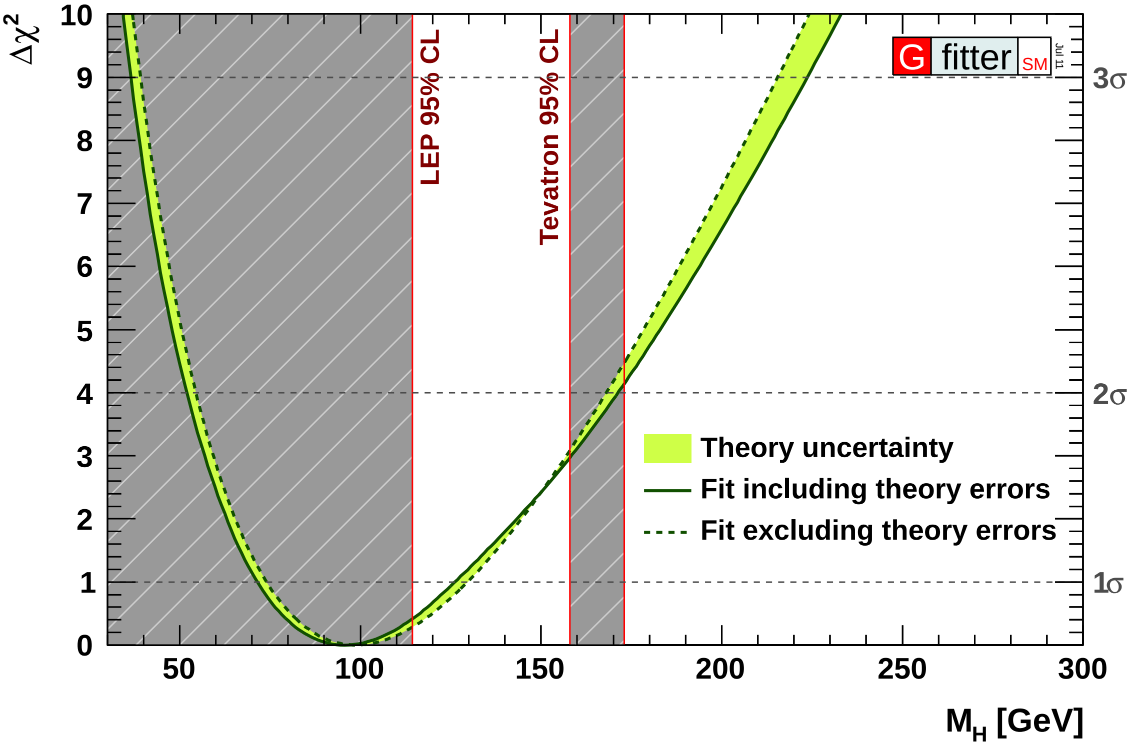


Fig. 24: SM precision tests favor the existence of a light Higgs. Taken from GFitter webpage [50].

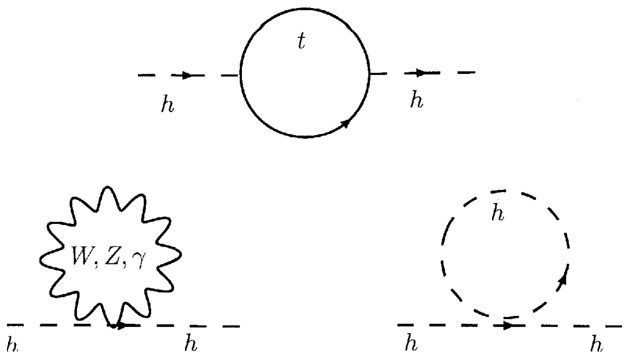


Fig. 25: Quadratic divergences to the Higgs mass in the SM, leading to the *hierarchy problem*.

In the case of the Standard Model, Dirac neutrino masses are generated, as for the quarks and leptons, through the Higgs mechanism and require the existence of right-haneded neutrinos. They also respect the leptonic number.

On the other hand, we can construct Majorana masses with the help of the charge conjugation matrix C

$$\mathcal{L}_{\text{Majorana mass}} = -\frac{M}{2}\left(\bar{\Psi}^c\Psi + \text{h.c.}\right) = -\frac{M}{2}\left(\Psi^T C\Psi + \text{h.c.}\right) \ . \tag{487}$$

In the frequently-used notation

$$\Psi = \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix} \ , \tag{488}$$

where ψ_1 and ψ_2 are two-component spinors, very convenient to use in case of Majorana and Weyl

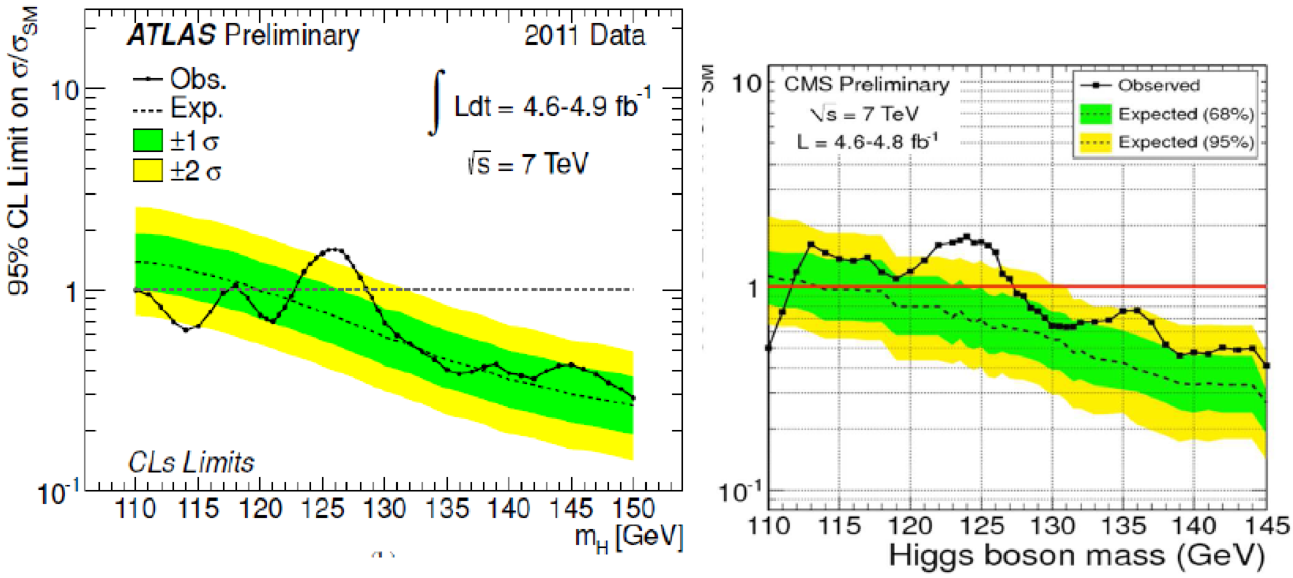


Fig. 26: Possible evidence of a Higgs scalar with mass around 125 GeV from ATLAS (left) and CMS (right), presented at the 2012 Moriond conference [54].

fermions, the Dirac and Majorana masses are given by

$$\begin{aligned}\mathcal{L}_{\text{Dirac mass}} &= -m_D\bar{\Psi}\Psi = -m_D\left(\psi_1\psi_2 + \bar{\psi}_1\bar{\psi}_2\right) \ ,\\ \mathcal{L}_{\text{Majorana mass}} &= -\frac{M}{2}\left(\bar{\Psi}^c\Psi + \text{h.c.}\right) = -\frac{M}{2}\left[\psi_1(i\tau^2)\psi_1 + \bar{\psi}_2(-i\tau^2)\bar{\psi}_2 + \text{h.c.}\right] \ . \end{aligned} \tag{489}$$

Majorana fermions N are defined by the Majorana condition $N = N^c$, for which the four-component fermion is

$$N = \begin{pmatrix} \psi \\ -i\tau^2\overline{\psi^T} \end{pmatrix} \ . \tag{490}$$

In the case of Standard Model (active) neutrinos, Majorana masses are not gauge invariant, but can be generated after the Higgs mechanism from the dimension-five operators

$$\mathcal{L}_{\text{Majorana mass}} = -\frac{1}{M}h_{ij}(\epsilon_{\alpha\beta}l_i^\alpha\Phi^\beta)^T C(\epsilon_{\gamma\delta}l_j^\gamma\Phi^\delta) = -\frac{1}{M}h_{ij}(\nu_i^\alpha\Phi^0 - e_i\Phi^+)^T C(\nu_j^\alpha\Phi^0 - e_j\Phi^+) \ . \tag{491}$$

The operator (491), called usually Weinberg operator, is now gauge invariant and after electroweak symmetry breaking it generate neutrino mass matrices

$$M_{ij}^\nu = \frac{v^2}{M}h_{ij} \ . \tag{492}$$

Data on neutrino oscillations (see later) and cosmology favor small masses, typically of order 10^{-2} eV. In this case, typical values (for Yukawas h's of order one) of the high mass scale in (491),(492) are of order $M \sim 10^{15}$ GeV, which, interestingly enough are close to the energy scale where gauge couplings tend to unify. It is often said that neutrino masses are maybe the first hint of a new physics at a very high scale M. Majorana masses violate the leptonic number by two units $\Delta L = 2$, which is potentially observable in neutrinoless double beta decay experiments.

An elegant explanation of the appearance of the Weinberg operator is *the seesaw mechanism* [55], in which the relevant fermions are the leptonic doublets $l_i =$ and three sterile Majorana neutrinos $N_i =$

N_i^c. The relevant lagrangian for neutrino masses is

$$\mathcal{L}_{\text{mass}} = -\sqrt{2}\,(\lambda_{ij}\bar{N}_j l_i \Phi + \text{h.c.}) \; - \; \frac{1}{2} M_{ij} N_i^T C N_j \; . \tag{493}$$

After electroweak symmetry breaking, we obtain the mass terms

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2} \begin{pmatrix} \nu_i^T & \psi_i^T \end{pmatrix} \begin{pmatrix} 0 & \lambda_{ij} v \\ \lambda_{ij} v & M_{ij} \end{pmatrix} \begin{pmatrix} \nu_j \\ \psi_j \end{pmatrix} + \text{h.c.}\, . \tag{494}$$

In what follows we define the Dirac mass matrix $m_{ij}^D = \lambda_{ij} v$. Let us consider for simplicity the one-generation case, in which case the mass matrix has the two eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left[M \pm \sqrt{M^2 + 4m^2} \right] \; . \tag{495}$$

For $m << M$, the lightest (heaviest) eigenvalue is $\lambda_l \simeq -m^2/M$ ($\lambda_h \simeq M$). For the 3×3 case, we obtain a symmetric mass matrix for the three light active neutrinos

$$M^\nu = \lambda \frac{v^2}{M} \lambda \; . \tag{496}$$

The result can be interpreted as the generation of an effective operator after integrating-out the heavy right-handed neutrinos, leading precisely to the Weinberg operator

$$\mathcal{L}_{\text{Majorana mass}} \;=\; - \,(\lambda \frac{1}{M} \lambda)_{ij}\; (l_i \Phi)^T C (l_j \Phi) \; . \tag{497}$$

The resulting neutrino mass matrix (496) can be diagonalized by a unitary transformation U called MNSP matrix

$$U^T m^\nu U = m^\nu_{\text{diag}} \; , \tag{498}$$

which can be parametrized as $U = VK$, where

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} , \; K = diag \begin{pmatrix} 1 & e^{i\phi_1} & e^{i(\phi_2+\delta)} \end{pmatrix} \; , \tag{499}$$

where $c_{12} \equiv \cos\theta_{12}$, etc.

11.2 Neutrino oscillations

One of the long standing puzzles in particles physics was the deficit of neutrino fluxes in reactors. The sun generates a large electron neutrino flux, produced from its various thermonuclear reactions. Its estimate was a factor of three larger than the measured electron neutrino flux in (Super)Kamiokande, in Japan. This puzzle defined the so-called solar neutrino flux deficit. On the other hand, muon neutrinos are produced in the earth atmosphere. Their measures flux was also less than the predicted value, leading to the so-called atmospheric neutrino flux deficit. In both cases, the correct explanation was the oscillation of neutrinos between different flavors. For the solar neutrino problem, the conversion is $\nu_e \to \nu_\nu$, whereas for the atmospheric neutrinos, the relevant conversion is $\nu_\mu \to \nu_\tau$. We present in what follows the field-theory description of this phenomenon. The relevant neutrino lagrangian is

$$\mathcal{L} = \bar{\nu}_i i \partial\!\!\!/ \nu_i - (\nu_i^T M_{ij}^\nu C \nu_j + \text{h.c.}) \; . \tag{500}$$

Diagonalization of the neutrino mass matrix proceeds through a unitary transformation

$$\nu_i = U_{ij} \nu'_j \; , \tag{501}$$

where ν'_j are mass eigenstates, whose wavefunction evolves according to the Schrodinger equation $|\nu'_i(t)\rangle = e^{-iE_i}|\nu'_i\rangle$. The probability of oscillation from flavor f to flavor f' after a propagating time t can then be computed to be

$$P_{f\to f'}(t) = |\langle \nu_{f'}|\nu_f(t)\rangle|^2 = \sum_i |U_{fi}U_{f'i}|^2 + 2\sum_{i>j} Re\{U_{fi}U^*_{f'i}U^*_{fj}U_{f'j}e^{i(E_j-E_i)t}\} , \qquad (502)$$

whereas the survival probability of the neutrino flavor f equals

$$P_{f\to f}(t) = |\langle \nu_f|\nu_f(t)\rangle|^2 = |\sum_i |U_{fi}|^2 e^{iE_it}|^2 . \qquad (503)$$

The energy difference can be estimated in the ultra-relativistic approximation to be $E_j - E_i \simeq (m_j^2 - m_i^2)/2p$. Therefore neutrino oscillation data only constrain mass square differences and not the absolute value of neutrino masses[12]. The current experimental data on neutrino masses give the numerical values

$$\begin{aligned} &\Delta m^2_{\rm atm} \sim 2.4\times 10^{-3} eV^2 \qquad (\nu_\mu - \nu_\tau) , \\ &\Delta m^2_{\rm sol} \sim 7.9\times 10^{-5} eV^2 \qquad (\nu_e - \nu_\mu) . \end{aligned} \qquad (504)$$

Adding also the data on the oscillations experiments we find

$$\begin{aligned} &\Delta m_{12} \simeq 0.008\, eV \quad , \quad \Delta m_{23} \simeq 0.03 - 0.07\, eV , \\ &\sin^2\theta_{12} \simeq 0.31 \ , \ 0.29 \le \sin^2\theta_{23} \le 0.71 \ , \ \sin^2\theta_{13} \simeq 0.023 . \end{aligned} \qquad (505)$$

12 Epilogue : Can Standard Model be the final theory ?

Most people believe that Standard Model is just an effective description, for a lot of various reasons :

- There are no *neutrino masses* at the renormalizable level in the Standard Model. The neutrino masses and mixings are often considered as a first hint towards a new mass scale beyond the Standard Model. The seesaw mechanism points towards heavy Majorana singlet neutrinos, maybe remnants of Grand Unified Theories.
- The mysterious *hierarchies* in the quarks/lepton masses and mixings. It is likely that quarks and leptons hierarchies hide the existence of new flavor symmetries or of a geometrical origin related to wave functions profiles in a higher-dimensional space.
- Standard Model has no viable *Dark Matter* candidate. This is currently maybe the most pressing problem: understanding the origin and the properties of the dark matter candidate, which provides about 30% of the energy density of the Universe.
- The problem with the *radiative stability* of the electroweak scale ("the hierarchy problem").
- SM has no accurate *gauge coupling unification*.

The last three problems find together a nice solution in low-energy supersymmetry. The elegant embedding of quarks and leptons into complete representations of $SU(5)$ also point out towards a unified gauge group structure.

- The *strong CP problem.* The most popular solution postulates the existence of new light particles, the axions, which exist in all string theories and often play a central role in their quantum consistency.
- *Gravity is not incorporated* into a renormalizable framework. The only viable well-studied framework of quantum gravity to date is string theory.
- The *cosmological constant* problem $\Lambda \sim 10^{-4}\ eV^4 \sim\ 10^{-120}\ M_P^4$. This is certainly the biggest mistery in modern physics.

On the other hand, any theory describing nature has to be validated by experiments. For the time being, LHC found no signal of new physics, but it completed the Standard Model pircture by

[12]Cosmological data sets however an upper bound on the sum of the neutrino masses of the order $\sum_i m_i \le 1$ eV.

discovering the Higgs boson with a mass around 125 GeV. It is still early to judge the viability of low-energy supersymmetry or extra dimensional models. One direction to follow is the couplings of the SM higgs to fermions, gauge bosons and to itself. If these couplings will show deviations from SM expectations, it will imply the existence of new forces, particles or resonances at TeV energies. In this case, LHC or future colliders should be able to see them after a couple of years of running. On the other hand, there are various mild indications of deviations from flavor universality in meson decays. If confirmed, due to the high sensitivity of flavor observables to new physics, they could be interpreted as the existence of new flavor violating processes induced by new physics in the multi TeV range energies.

In any case, nature will likely reserve us surprises which will challenge our view of fundamental interactions and symmetries at high energies. It will be your challenge and duty to encover them and continue our fascinating journey towards the understanding the microscopic laws of our mysterious universe.

Acknowledgments

I would like to thank the organizers of the CERN summer school 2011 for providing a stimulating and very pleasant environment. This work was partially supported by the European ERC Advanced Grant 226371 MassTeV, by the European Initial Training Network PITN-GA-2009-237920 and by the Agence Nationale de la Recherche.

Appendices

A Path integral quantization

A.1 Path integral quantization in quantum mechanics

Let us consider a quantum mechanical system of coordinates and momenta $\mathbf{q}, \mathbf{p}$, of hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}) \ . \tag{A.1}$$

The transition amplitude in quantum mechanics can be written as a sum over all possible paths, with a weight provided by the classical action

$$\langle \mathbf{q}_2, T_2 | \mathbf{q}_1, T_1 \rangle \equiv \langle \mathbf{q}_2 | e^{-\frac{iH(T_2-T_1)}{\hbar}} | \mathbf{q}_1 \rangle \ = \ \int \mathcal{D}\mathbf{q}(t) \ e^{\frac{i}{\hbar} S[\mathbf{q}(t)]} \ , \tag{A.2}$$

where the integral is over all paths $\mathbf{q}(t)$ with the boundary conditions $\mathbf{q}(t_1) = \mathbf{q}_1, \mathbf{q}(t_2) = \mathbf{q}_2$, and where the classical action is

$$S[\mathbf{q}(t)] = \int_{T_1}^{T_2} dt \ L[\mathbf{q}(t)] = \int_{T_1}^{T_2} dt \left[\frac{m}{2} \dot{\mathbf{q}}^2 - V(\mathbf{q}) \right] \ . \tag{A.3}$$

The classical action that satisfies $\frac{\delta S[\mathbf{q}(t)]}{\delta \mathbf{q}(t)} = 0$ contributes most to the transition amplitude, but all other classically forbidden paths contribute in a precised way. The natural question is of course what is the measure one is integrating upon. A mathematically heuristic, but intuitively clear definition is of discretizing the time propagation into small time intervals

$$\begin{aligned} & t_0 = T_1 \ , \ t_1 \ , \ t_2 \ , \ \cdots t_N = T_2 \quad , \quad \text{with} \ \ t_i - t_{i-1} = \epsilon \ \ \text{and} \ \ N\epsilon = T \ , \\ & \text{and} \ \ \mathbf{q}(t_i) \equiv \mathbf{q}_i \ , \ \mathbf{q}(T_1) = \mathbf{q}_1 \ , \ \mathbf{q}(T_2) = \mathbf{q}_2 \ . \end{aligned} \tag{A.4}$$

The path integral is then an (appropriately normalized) ordinary integrals over the intermediate coordinates $\mathbf{q}_k$,

$$\int \mathcal{D}\mathbf{q}(t) \ \sim \ \prod_k \int d\mathbf{q}_k \ . \tag{A.5}$$

The next step is to split the evolution propagator into the small intermediate time propagations

$$e^{-\frac{iH(T_2-T_1)}{\hbar}} = \prod_i e^{-\frac{iH\epsilon}{\hbar}} \simeq \prod_i (1 - \frac{iH\epsilon}{\hbar}) \tag{A.6}$$

and introduce an intermediate set of states in each of the propagation time intervals ϵ

$$1 = \prod_i \int dq_k^i \, |\mathbf{q}_k\rangle\langle\mathbf{q}_k| \,. \tag{A.7}$$

By the superposition principle, the transition amplitude can therefore be written as

$$\langle\mathbf{q}_2, T_2|\mathbf{q}_1, T_1\rangle = \lim_{N\to\infty} \int \prod_{q^i=1}^{N-1} d\mathbf{q}_p \prod_{k^i=0}^{N-1} \langle\mathbf{q}_{k+1}|e^{-\frac{iH\epsilon}{\hbar}}|\mathbf{q}_k\rangle \,. \tag{A.8}$$

The transition amplitude involves then matrix elements of the type $\langle\mathbf{q}_{k+1}|e^{-\frac{iH\epsilon}{\hbar}}|\mathbf{q}_k\rangle$. For an appropriate order (called Weyl ordering) of the operators in the hamiltonian, the result is (we set $\hbar = 1$ from now on)

$$\langle\mathbf{q}_{k+1}|e^{-iH\epsilon}|\mathbf{q}_k\rangle = \prod_i \int \frac{dp_k^i}{2\pi} e^{-i\epsilon H(\frac{\mathbf{q}_{k+1}+\mathbf{q}_k}{2}, p_k^i) + i\sum_i p_k^i (q_{k+1}^i - q_k^i)} \tag{A.9}$$

and the whole transition amplitude becomes

$$\langle\mathbf{q}_2, T_2|\mathbf{q}_1, T_1\rangle \;=\; \prod_{i,k} \int dq_k^i \int \frac{dp_k^i}{2\pi} \; e^{i\sum_k \left(\sum_i p_k^i (q_{k+1}^i - q_k^i) - \epsilon H(\frac{\mathbf{q}_{k+1}+\mathbf{q}_k}{2}, p_k^i)\right)} \,. \tag{A.10}$$

Notice that this is the discretized version of the path integral expression

$$\langle\mathbf{q}_2, T_2|\mathbf{q}_1, T_1\rangle \;=\; \int \mathcal{D}\mathbf{q}(t) \int \mathcal{D}\mathbf{p}(t) \;\; e^{i\int_{T_1}^{T_2} dt \; (\mathbf{p}\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p}))} \,. \tag{A.11}$$

Coming back to the discretized form, the integral over momenta is gaussian and can be carried out explicitly, leading to the final explicit expression

$$\langle\mathbf{q}_2, T_2|\mathbf{q}_1, T_1\rangle \;=\; \frac{1}{\mathcal{N}(\epsilon)} \prod_k \int \frac{d\mathbf{q}_k}{\mathcal{N}(\epsilon)} \; e^{i\sum_k \left(\frac{m(\mathbf{q}_{k+1}-\mathbf{q}_k)^2}{2\epsilon} - \epsilon V(\frac{\mathbf{q}_{k+1}+\mathbf{q}_k}{2}\right)} \,, \tag{A.12}$$

where $\mathcal{N}(\epsilon) = \sqrt{\frac{2\pi\epsilon}{-im}}$. Further simplification takes place by considering periodic boundary conditions $\mathbf{q}(T_1) = \mathbf{q}(T_2) = \mathbf{q}_0$ and integrating over $\mathbf{q}_0$

$$Z = \int d\mathbf{q}_0 \; \langle\mathbf{q}_0|e^{-iH(T_2-T_1)}|\mathbf{q}_0\rangle \;=\; \int d\mathbf{q}_0 \int \mathcal{D}\mathbf{q}(t) \; e^{\frac{i}{\hbar} S[\mathbf{q}(t)]} \,. \tag{A.13}$$

Indeed, by inserting a complete set of states $|n\rangle$, it can easily be shown that

$$Z = \sum_n e^{-iE_n \Delta T} \,. \tag{A.14}$$

Formulae (A.13,A.14) allow often for the determination of the quantum mechanicsl energy levels E_n, by computing the "partition function " Z under suitable approximations. The analogy of Z with the partition function in statistical physics is clearly seen by Wick rotation to euclidian time $\Delta T = i\beta$. In this case, one gets the path integral formulae for the partition function of a quantum system

$$Z(\beta) \;=\; \sum_n e^{-\beta E_n} \;=\; \int d\mathbf{q}_0 \int \mathcal{D}\mathbf{q}(t) \; e^{-\beta S[\mathbf{q}(t)]} \,, \tag{A.15}$$

supplemented with the boundary condition $\mathbf{q}(i\beta) = \mathbf{q}(0) = \mathbf{q}_0$.

In practice we are interesting in quantum transition amplitudes from initial to final hamiltonian eigenstates in a collision or transition, cross sections or particle lifetimes, instead of the amplitude discussed above. The transition from an initial state $|\psi_i\rangle$ at a remote time (where is no interaction) $t = -T/2$ to a final one $|\psi_i\rangle$ at a future time $t = +T/2$ is given by

$$A_{fi} = \langle \psi_f(T/2)|\psi_i(-T/2)\rangle = \langle \psi_f|e^{-iHT}|\psi_i\rangle = \int dq dq' \ \langle \psi_f|q'\rangle\langle q'|e^{-iHT}|q\rangle\langle q|\psi_i\rangle \ , \qquad \text{(A.16)}$$

Using (A.11), one can express the transition amplitude as

$$A_{fi} = \int dq dq' \ \psi_f^*(q') \int_{q(-T/2)=q, q(T/2)=q'} \mathcal{D}\mathbf{q}(t) \int \mathcal{D}\mathbf{p}(t) \ e^{i\int_{-T/2}^{T/2} dt \ (\mathbf{p}\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p}))} \psi_i(q) \ . \qquad \text{(A.17)}$$

This is actually equivalent to perform a unconstrained functional integral over $q(t)$, with the arguments of the initial (final) wavefunctions evaluated at the initial (final) time

$$A_{fi} = \int \mathcal{D}\mathbf{q}(t) \int \mathcal{D}\mathbf{p}(t) \ \psi_f^*(q') \ e^{i\int_{-T/2}^{T/2} dt \ (\mathbf{p}\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p}))} \psi_i(q) \ . \qquad \text{(A.18)}$$

For a hamiltonian quadratic in momenta (with coefficient independent of q) as in the canonical case (A.1) and up to a irrelevant (for most practical purposes) multiplicative constant, this is equivalent to the lagrangian expression

$$A_{fi} = \int \mathcal{D}\mathbf{q}(t) \ \psi_f^*(q') \ e^{i\int_{-T/2}^{T/2} dt \ L(\mathbf{q},\dot{\mathbf{q}})} \psi_i(q) \ . \qquad \text{(A.19)}$$

A.2 Path integral quantization in field theory

The starting point in the path-integral formalism in field theory is the vacuum-to-vacuum amplitude in the presence of an external field

$$\langle 0, \text{out}|0, \text{in}\rangle_J \equiv Z(J) = e^{iW(J)} = \int \mathcal{D}\phi \ e^{i\int d^4x [\mathcal{L}(\phi) + J(x)\phi(x)]} \ . \qquad \text{(A.20)}$$

It can be shown that the time-ordered correlation functions, which appear in Feynman rules in perturbation theory, can be computed according to

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \int \mathcal{D}\phi \ \phi(x_1)\cdots\phi(x_n) \ e^{i\int d^4x [\mathcal{L}(\phi) + J(x)\phi(x)]} \ . \qquad \text{(A.21)}$$

The interest of introducing the vacuum-to-vacuum amplitudes is that they allow to compute Green functions via functional differentiation

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\cdots\left(\frac{1}{i}\frac{\delta}{\delta J(x_n)}\right) Z(J) \ ,$$
$$G^{(n)}(x_1\cdots x_n) = \langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle_c = \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)} W(J) \ , \qquad \text{(A.22)}$$

where $G^{(n)}(x_1\cdots x_n)$ are *connected* n-point Green functions. $Z(J)$ can be computed in perturbation theory by separating the free lagrangian from the interation

$$Z(J) = \int \mathcal{D}\phi \ e^{i\int d^4x [\mathcal{L}_0(\phi) + \mathcal{L}_{\text{int}}(\phi) + J(x)\phi(x)]} = e^{i\int d^4x \mathcal{L}_{\text{int}} \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} Z_0(J) \ , \qquad \text{(A.23)}$$

where

$$Z_0(J) = \int \mathcal{D}\phi \ e^{i\int d^4x [\mathcal{L}_0(\phi) + J(x)\phi(x)]} = e^{\frac{1}{2}\int d^4x d^4y J(x) D_F(x-y) J(y)} \ , \qquad \text{(A.24)}$$

where $D_F(x-y)$ is the Feynman propagator, in the case of scalar fields given by (65). The eqs. (A.22), (A.23),(A.24) define perturbation theory and Feynman diagrams.

A.3 Path integral quantization in non-abelian gauge theories: the Faddeev-Popov lagrangian

The naive vacuum-to-vacuum amplitude for zero external source in nonabelian gauge theories is

$$Z(0) \;=\; \int \mathcal{D}A \; e^{i\int d^4x\,(-\frac{1}{4}F^{a,2}_{mn})} \;, \tag{A.25}$$

where $\mathcal{D}A$ is the measure over all possible gauge fields configurations (Lorentz and internal indices are suppressed for simplicity). Things are not so simple in this case, however. Even in the abelian case, the quadratic lagrangian has no inverse due to the gauge invariance. The same is true for the non-abelian case, with a further subtlety that we will encounter. In order to quantize the theory, one has to choose a gauge-fixing condition, $\mathcal{F}(A) = 0$. One can insert then in the path-integral (A.25) the following identity

$$1 = \int \mathcal{D}\alpha(x)\; \delta(\mathcal{F}(A^\alpha))\; \det\left(\frac{\delta \mathcal{F}(A^\alpha)}{\delta\alpha}\right) \;, \tag{A.26}$$

where

$$A^\alpha_m = U(\alpha)(A_m + \frac{i}{g}\partial_m)U^{-1}(\alpha) \quad \to \quad (A^a_m)^\alpha \simeq A^a_m + \frac{i}{g}D_m\alpha^a \;, \tag{A.27}$$

is the gauge transform of the gauge field $A_m = A^a_m T_a$, with T_a generators of the corresponding gauge field Lie group. The last line in (A.27) refers to the infinitesimal transformations, of parameters α^a and $D_m\alpha^a = \partial_m\alpha^a + gf^{abc}A^b_m\alpha^c$.

By gauge invariance $S[A] \equiv -\frac{1}{4}F^{a,2}_{mn} = S[A^\alpha]$. The path-integral measure is also invariant to the replacement $A \to A^\alpha$. Then one can replace $Z(0)$ with

$$Z(0) \;=\; \int \mathcal{D}\alpha \int \mathcal{D}A\; e^{iS[A]}\; \delta(\mathcal{F}(A))\; \det\left(\frac{\delta\mathcal{F}(A^\alpha)}{\delta\alpha}\right) \;. \tag{A.28}$$

Let us consider covariant gauge fixing conditions of the form $\mathcal{F}^a(A) = \partial^m A^a_m - \omega^a(x)$. Since physical observables should not depend on changes in the gauge fixing conditions, one cold just integrate over ω with a gaussian weight

$$\int \mathcal{D}\omega\; e^{-i\int d^4x \frac{\omega_a^2(x)}{2\xi}}\; \delta(\partial^m A^a_m - \omega_a(x)) \;=\; N(\xi)\; e^{-i\int d^4x \frac{1}{2\xi}(\partial^m A^a_m)^2} \;, \tag{A.29}$$

with $N(\xi)$ an irrelevant constant that will cancel out in physical computations. For Yang-Mills theories, one can find

$$\frac{\delta\mathcal{F}(A^\alpha)}{\delta\alpha} = \frac{1}{g}\partial^m D_m \quad , \quad \det(\frac{1}{g}\partial^m D_m) = \int \mathcal{D}(c_a, \tilde{c}_a)\; e^{i\int d^4x\; \tilde{c}_a\,(-\partial^m D_m c_a)} \;, \tag{A.30}$$

where $c_a, \tilde{c}_a$ are scalar fields in the adjoint representation of the gauge group, but are anticommuting ! They therefore violate the spin-statistics theorem and cannot be physical; for this reason they are called *Faddeev-Popov ghosts* [16]. The integrand in (A.28) is independent on the gauge group elements α and therefore the corresponding integral gives an irrelevant (though infinite) constant which will cancel upon dividing the various correlation functions by the vacuum-to-vacuum amplitudes, i.e. considering connected diagrams. The final generating functional takes therefore the form

$$Z(0) \;=\; \int \mathcal{D}(A, c, \tilde{c})\; e^{i\int d^4x \mathcal{L}} \;, \tag{A.31}$$

where

$$\mathcal{L} \;=\; -\frac{1}{4}(F^a_{mn})^2 - \frac{1}{2\xi}(\partial^m A^a_m)^2 + \tilde{c}_a(-\partial^m D_m c_a) \;. \tag{A.32}$$

Since the ghosts are unphysical, they cannot be propagating asymptotic states. This can be proven with the help of the BRST symmetry [56], but this it is beyond the scope of these notes. The lagrangian (A.32) sets the ground for the Lorentz-covariant perturbation theory and Feynman diagrams displayed in Section 4.4.

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