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Lecture Notes on Extra Dimensions

-Kaluza-Klein compactification : one extra dimension on a circle

-Generalizations, orbifolds, symmetry breaking by compactification

-Branes in extra dimensions

Kaluza-Klein compactification : one extra dimension on a circle

X-dimensions

unobservable $\rightarrow$ finite size

detection : KK spectrum

ex: 1 extra dim on a circle $\quad y \equiv y + 2\pi R$

$$\Rightarrow \phi(x^\mu, y) = \sum_{n \in \mathbb{Z}} \phi_n(x)\, e^{i y n / R}$$

$\Box_5 \phi = M^2 \phi$ $\qquad$ signature : $(- + + \ldots +)$

massive scalar field in 5d

$$\Box_5 = \Box_4 + \partial_y^2$$

$$\Rightarrow \left(\Box - \frac{n^2}{R^2}\right)\phi_n = M^2 \phi_n \Rightarrow \Box \phi_n = \underbrace{\left(M^2 + \frac{n^2}{R^2}\right)}_{\text{KK spectrum}} \phi_n$$

Effective action : $S = \frac{1}{2}\int d^4x \int_0^{2\pi R} dy \left\{ -(\partial_M \phi)^2 + M^2 \phi^2 \right\}$

$$\int dy\, e^{i(m+n)y} = 2\pi R\, \delta_{m+n,0}$$

$$\Rightarrow S = \sum_n \frac{1}{2}\int d^4x\, 2\pi R \left\{ -(\partial_\mu \phi_n)^2 - \underbrace{\frac{nm}{R^2}\phi_n \phi_m \delta_{n+m}}_{\frac{n^2}{R^2}\phi_n^2} + M^2 \phi_n^2 \right\}$$

$$= \frac{2\pi R}{2}\int d^4x \sum_n \left\{ -(\partial_\mu \phi_n)^2 + \left(M^2 + \frac{n^2}{R^2}\right)\phi_n^2 \right\}$$

normalized field: $\tilde{\phi}_n = \dfrac{\phi_n}{\sqrt{2\pi R}}$

Unification of forces from 5d (KK idea)

1) Gauge field $\Rightarrow$ gauge + scalar

$$M = (\mu, 4) \longleftrightarrow (x^\mu, y) \qquad y \equiv x^4$$

$$A_M = \begin{pmatrix} A_\mu(x,y) \\ \phi(x,y) \end{pmatrix} \qquad \phi \equiv A_4$$

y-dependence : expand in series as before

$$F_{MN}^2 = F_{\mu\nu}^2 + 2 F_{\mu 4}^2 \qquad F_{\mu 4} = \partial_\mu A_4 - \partial_4 A_\mu = \partial_\mu \phi - \partial_y A_\mu$$

5d gauge invariance : $\delta A_M = \partial_M \Lambda(x,y)$

$$\Rightarrow \delta A_\mu = \partial_\mu \Lambda \quad , \quad \delta\phi = \partial_y \Lambda$$

$$\Rightarrow \delta A_\mu^n = \partial_\mu \Lambda_n \quad , \quad \delta \phi_n = i\frac{n}{R} \Lambda_n$$

- $n=0 \Rightarrow$ usual 4d gauge invariance
- $n \neq 0 \Rightarrow$ gauge choice : $\phi_n = 0$ (A_μ^n massive)

longitudinal polarization

$$\Rightarrow F_{\mu 4}^n = \begin{cases} \frac{in}{R} A_\mu^n & n \neq 0 \\ \partial_\mu \phi_0 & n=0 \end{cases}$$

$$\Rightarrow \int dy \, \frac{1}{4} F_{MN}^2 \underset{2\pi R}{=} \frac{1}{4} F_{\mu\nu}^{(n)2} + \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2}\frac{n^2}{R^2} A_\mu^{(n)2}$$

$\Rightarrow$ 4d: U(1) gauge field + massless scalar + massive KK-vectors $n = \pm 1, \pm 2, \dots$
with masses^2 $(n/R)^2$

2) gravity 5d $\to$ gravity 4d + U(1) + scalar (radion)

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\mu 4} \\ G_{\mu 4} & G_{44} \end{pmatrix}(x,y)$$

$$\delta G_{MN} = G_{MP}\,\partial_N \xi^P + G_{NP}\,\partial_M \xi^P + \xi^P \partial_P G_{MN} \qquad \xi^M = \begin{pmatrix} \xi^\mu \\ \xi^4 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \delta G_{44} = 2 G_{4\mu}\partial_4 \xi^\mu + 2 G_{44}\partial_4 \xi^4 + \xi^\mu \partial_\mu G_{44} + \xi^4 \partial_4 G_{44} \\ \delta G_{\mu 4} = G_{\mu\nu}\partial_4 \xi^\nu + G_{\mu 4}\partial_4 \xi^4 + G_{4\nu}\partial_\mu \xi^\nu + G_{44}\partial_\mu \xi^4 \\ \qquad\qquad + \xi^\nu \partial_\nu G_{\mu 4} + \xi^4 \partial_4 G_{\mu 4} \end{cases}$$

δG_{44} : $\xi^\mu = 0 \Rightarrow \delta G_{44} = 2 G_{44}\,\partial_4 \xi^4 + \xi^4 \partial_4 G_{44}$

$$\delta G_{44}^n = \sum_m \left\{ 2 G_{44}^{n-m} \frac{im}{R} \xi^4_m + \frac{i(n-m)}{R} \xi^4_m G_{44}^{n-m} \right\}$$

$$= \sum_m \frac{i(n+m)}{R} \xi^4_m G_{44}^{n-m}$$

$G_{44} = e^{2\phi} \Rightarrow \delta\phi = \partial_y \xi^4 + \xi^4 \partial_y \phi$

$$\Rightarrow \delta\phi_n = \frac{in}{R}\xi^4_n + \sum_m \frac{im}{R}\phi_m \xi^4_{n-m}$$

non-trivial ξ^4_n, $n \neq 0$ gauge choice : $\phi_n = 0$ (or $G^n_{44} = 0$)

$\delta G_{\mu 4}$: $\xi^4 = 0 \Rightarrow \delta G_{\mu 4} = G_{\mu\nu}\partial_4 \xi^\nu + G_{4\nu}\partial_\mu \xi^\nu + \xi^\nu \partial_\nu G_{\mu 4}$

non-trivial ξ^μ_n, $n \neq 0$ gauge choice : $G^n_{\mu 4} = 0$

$\Rightarrow G^n_{\mu\nu}$: massive spin-2 (5 helicities : ± 2, $\underset{G^n_{\mu 4}}{\pm 1}$, $\underset{G^n_{44}}{\underset{\phi}{0}}$)

with gauge choice ⇒

$$G_{MN}(x,y) = e^{2\phi(x)} \begin{pmatrix} \hat{G}_{\mu\nu}(x,y) & A_\mu(x) \\ A_\nu(x) & 1 \end{pmatrix} \qquad G_{44} = e^{2\phi}, \quad G_{\mu 4} = e^{2\phi} A_\mu$$

leftover gauge freedom $\xi^\mu(x)$, $\xi^4(x)$ $\qquad \delta x^\mu = -\xi^\mu$, $\delta y = -\xi^4$

$$\delta \hat{G}_{\mu\nu} = \delta\left(e^{-2\phi} G_{\mu\nu}\right) =$$

$$\delta G_{44} = \xi^\mu \partial_\mu G_{44} \Rightarrow \boxed{\delta\phi = \xi^\mu \partial_\mu \phi}$$

$$\delta G_{\mu 4} = G_{4\nu}\partial_\mu \xi^\nu + G_{44}\partial_\mu \xi^4 + \xi^\nu \partial_\nu G_{\mu 4} \Rightarrow$$

$$\delta A_\mu = \delta\left(e^{-2\phi} G_{\mu 4}\right) = e^{-2\phi}\delta G_{\mu 4} - 2 e^{-2\phi} G_{\mu 4}\delta\phi$$

$$= A_\nu \partial_\mu \xi^\nu + \partial_\mu \xi^4 + \underbrace{e^{-2\phi}\xi^\nu \partial_\nu\left(e^{2\phi}A_\mu\right)}_{\xi^\nu \partial_\nu A_\mu + \cancel{2A_\mu \xi^\nu \partial_\nu \phi}} - \cancel{2A_\mu \xi^\nu \partial_\nu \phi}$$

$$\Rightarrow \boxed{\delta A_\mu = A_\nu \partial_\mu \xi^\nu + \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^4}$$

$$\delta G_{\mu\nu} = G_{\mu\rho}\partial_\nu \xi^\rho + G_{\nu\rho}\partial_\mu \xi^\rho + \xi^\rho \partial_\rho G_{\mu\nu}$$

$$= G_{(\mu\lambda}\partial_{\nu)}\xi^\lambda + \xi^\lambda \partial_\lambda G_{\mu\nu} + G_{(\mu 4}\partial_{\nu)}\xi^4 + \xi^4 \partial_4 G_{\mu\nu}$$

$$\Rightarrow \delta\hat{G}_{\mu\nu} = e^{-2\phi}\delta G_{\mu\nu} - 2e^{-2\phi} G_{\mu\nu}\delta\phi$$

$$= \hat{G}_{(\mu\lambda}\partial_{\nu)}\xi^\lambda + \xi^\lambda \partial_\lambda \hat{G}_{\mu\nu} + \cancel{2(\xi^\lambda \partial_\lambda \phi)\hat{G}_{\mu\nu}} - \cancel{2\hat{G}_{\mu\nu}\xi^\lambda \partial_\lambda \phi}$$

$$+ A_{(\mu}\partial_{\nu)}\xi^4 + \xi^4 \partial_4 \hat{G}_{\mu\nu}$$

$$\Rightarrow \boxed{\delta\hat{G}_{\mu\nu} = \hat{G}_{(\mu\lambda}\partial_{\nu)}\xi^\lambda + \xi^\lambda \partial_\lambda \hat{G}_{\mu\nu} + A_{(\mu}\partial_{\nu)}\xi^4 + \xi^4 \partial_4 \hat{G}_{\mu\nu}} \qquad [\,(\mu,\nu) = \mu\nu + \nu\mu\,]$$

⇒ $\xi^\mu(x)$: 4d diffeomorphisms , $\xi^4(x)$: 4d U(1) gauge transformations

(from 5d diffeo's)

$$ds^2 = e^{2\phi}\, d\hat{s}^2$$

$$d\hat{s}^2 = \hat{G}_{\mu\nu}\, dx^\mu dx^\nu + 2A_\mu dx^\mu dy + dy\, dy$$

$$= \left(\hat{G}_{\mu\nu} - A_\mu A_\nu\right) dx^\mu dx^\nu + \left(dy + A_\mu dx^\mu\right)^2$$

$$\delta_4\left(dy + A_\mu dx^\mu\right) = -dx^\mu\, \partial_\mu \xi^4 + \partial_\mu \xi^4\, dx^\mu = 0$$

$$\delta_4\big(\underbrace{\hat{G}_{\mu\nu} - A_\mu A_\nu}_{\hat{g}_{\mu\nu}}\big) = \cancel{A_{(\mu}\partial_{\nu)}\xi^4} + \xi^4 \partial_4 \hat{G}_{\mu\nu} - \cancel{A_{(\mu}\partial_{\nu)}\xi^4} = \xi^4 \partial_4 \hat{G}_{\mu\nu} = 0$$

for $\hat{G}_{\mu\nu}$

$$\Rightarrow \quad G_{MN} = e^{2\phi} \begin{pmatrix} \overset{(x,0)}{\hat{g}_{\mu\nu}} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}$$

$$* \quad R\left[e^{2\phi} g_{\mu\nu}\right] = e^{-2\phi}\left\{R - 2(D-1) D^2\phi - (D-2)(D-1)(D\phi)^2\right\}$$

$$\sqrt{g}\left[e^{2\phi} g_{\mu\nu}\right] = e^{D\phi}\sqrt{g}$$

$$\Rightarrow \quad \sqrt{g}\, R^{(5)}\left[e^{2\phi}\hat{G}_{MN}\right] = e^{3\phi}\left\{\hat{R}^{(5)} - 8\hat{D}_\mu^2\phi - 12(\hat{D}_\mu\phi)^2\right\}$$

$$\left[\begin{array}{ll} R^\lambda{}_{\mu\nu\rho} = \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\rho\sigma} - \Gamma^\sigma_{\mu\rho}\Gamma^\lambda_{\nu\sigma} & R_{\mu\rho} = R^\lambda{}_{\mu\lambda\rho} \\ \Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\sigma\nu}\left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda}\right) & R = g^{\mu\rho} R^\lambda{}_{\mu\lambda\rho} \end{array}\right]$$

$$\hat{G} = \begin{pmatrix} \hat{g} + A \otimes A^T & A \\ A^T & 1 \end{pmatrix} = \begin{pmatrix} \vec{g}_1 + \vec{A} A_1 & \vec{g}_2 + \vec{A} A_2 & \cdots & \vec{g}_n + \vec{A} A_n & \vec{A} \\ A_1 & A_2 & & A_n & 1 \end{pmatrix}$$

$$\vec{g}_i = \begin{pmatrix} \hat{g}_{1i} \\ \hat{g}_{2i} \\ \vdots \\ \hat{g}_{ni} \end{pmatrix}$$

1) $$\det \begin{pmatrix} \vec{g}_1 + \vec{A} A_1 & \vec{g}_2 + \vec{A} A_2 & \cdots & \vec{g}_n + \vec{A} A_n & \vec{A} \\ A_1 & A_2 & & A_n & 1 \end{pmatrix} = \det \hat{G}$$

$$= \det \begin{pmatrix} \vec{g}_1 & \cdots & \vec{g}_n & \vec{A} \\ 0 & & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} \hat{g} & A \\ & 1 \end{pmatrix} = \det \hat{g}$$

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2) $$\det\left(\hat{g} + A \otimes A^T\right) = \det \hat{g} + \det\left(1 + \hat{g}^{-1} A \otimes A^T\right)$$

$$\det\left(1 + B \otimes A^T\right) = \det\left(\vec{e}_1 + A_1 \vec{B},\ \vec{e}_2 + A_2 \vec{B},\ \cdots\ \vec{e}_n + A_n \vec{B}\right)$$

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$ ← i-th line

$$= \det\left(\vec{e}_1 + A_1 \vec{B},\ \vec{e}_2 - \frac{A_2}{A_1}\vec{e}_1,\ \cdots,\ \vec{e}_n - \frac{A_n}{A_1}\vec{e}_1\right)$$

$$= \underbrace{\begin{vmatrix} 1 + A_1 B_1 & -A_2/A_1 & \cdots & -A_n/A_1 \\ A_1 B_2 & 1 & & 0 \\ \vdots & & & \vdots \\ A_1 B_n & 0 & & 1 \end{vmatrix}}_{M_n} = -(-)^n \frac{A_n}{A_1} \begin{vmatrix} A_1 B_2 & 1 & & \\ \vdots & & \ddots & \\ & & & 1 \\ A_1 B_n & & & 0 \end{vmatrix}$$

$$+ \underbrace{\begin{vmatrix} 1 + A_1 B_1 & \cdots & -\frac{A_{n-1}}{A_1} \\ \vdots & & \\ A_1 B_{n-1} & & 1 \end{vmatrix}}_{M_{n-1}}$$

$$\Rightarrow \det M_n = \det M_{n-1} - (-)^n \frac{A_n}{A_1} (-)^{n-1} A_1 B_n \det 1 = \det M_{n-1} + A_n B_n$$

$$= \cdots = \vec{A} \cdot \vec{B}$$

$$\Rightarrow \det\left(1+\hat{g}^{-1}A\otimes A^T\right) = A^T\hat{g}^{-1}A$$

$$\Rightarrow \det\left(\hat{g}+A\otimes A^T\right) = \det g + A^T\hat{g}^{-1}A$$

$$\Rightarrow \hat{G}^{-1} = \begin{pmatrix} \hat{g}^{-1} & -\hat{g}^{-1}A \\ -A^T g^{-1} & 1+A^T\hat{g}^{-1}A \end{pmatrix}$$

$$* \begin{pmatrix} \hat{g}+A\otimes A^T & A \\ A^T & 1 \end{pmatrix}\begin{pmatrix} \hat{g}^{-1} & -\hat{g}^{-1}A \\ -A^T\hat{g}^{-1} & 1+A^T\hat{g}^{-1}A \end{pmatrix} =$$

$$= \begin{pmatrix} 1+\cancel{A\otimes A^T\hat{g}^{-1}}-\cancel{A\otimes A^T\hat{g}^{-1}} & \cancel{-A}-\cancel{A\,A^T\hat{g}^{-1}A}+\cancel{A}+\cancel{A\,A^T\hat{g}^{-1}A} \\ \cancel{A^T\hat{g}^{-1}}-\cancel{A^T\hat{g}^{-1}} & -\cancel{A^T\hat{g}^{-1}A}+1+\cancel{A^T\hat{g}^{-1}A} \end{pmatrix} \quad \text{ok}$$

fields: y-independent

$$\Gamma^\sigma_{\mu\nu} = \hat{\Gamma}^\sigma_{\mu\nu} + \frac{1}{2}\hat{G}^{\sigma\nu}\left\{\partial_\mu(A_\lambda A_\nu)+\partial_\lambda(A_\mu A_\nu)-\partial_\nu(A_\mu A_\lambda)\right\} + \frac{1}{2}\hat{G}^{\sigma 4}\left(\partial_\mu A_\lambda+\partial_\lambda A_\mu-\cancel{\partial_y \hat{g}_{\mu\lambda}}\right)$$

$$= \hat{\Gamma}^\sigma_{\mu\nu} - \frac{1}{2}A^\sigma\left(\cancel{\partial_\mu A_\lambda+\partial_\lambda A_\mu}\right) + \frac{1}{2}\hat{g}^{\sigma\nu}\underbrace{\left(A_\nu\partial_\mu A_\lambda + A_\lambda\partial_\mu A_\nu + A_\mu\partial_\lambda A_\nu + A_\nu\partial_\lambda A_\mu - A_\mu\partial_\nu A_\lambda - A_\lambda\partial_\nu A_\mu\right)}_{A_\lambda F_{\mu\nu}+A_\mu F_{\lambda\nu}+\cancel{A_\nu(\partial_\mu A_\lambda+\partial_\lambda A_\mu)}}$$

$$\Rightarrow$$

$$\Gamma^\sigma_{\mu\lambda} = \hat{\Gamma}^\sigma_{\mu\lambda} + \frac{1}{2}\hat{g}^{\sigma\nu}\left(A_\lambda F_{\mu\nu}+A_\mu F_{\lambda\nu}\right)$$

$$\Gamma^\Sigma_{44} = \frac{1}{2}\hat{G}^{\Sigma N}\left(2\partial_4\hat{G}_{4N}-\partial_N\hat{G}_{44}\right) = 0$$

$$\Gamma^\Sigma_{4\mu} = \frac{1}{2}\hat{G}^{\Sigma N}\left(\cancel{\partial_4\hat{G}_{\mu N}}+\partial_\mu\hat{G}_{4N}-\partial_N\hat{G}_{\mu 4}\right) = \frac{1}{2}\hat{G}^{\Sigma\nu}F_{\mu\nu} \Rightarrow \begin{cases} \Gamma^\sigma_{4\mu} = \frac{1}{2}\hat{g}^{\sigma\nu}F_{\mu\nu} \\ \Gamma^4_{4\mu} = -\frac{1}{2}A^\nu F_{\mu\nu} \end{cases}$$

$$\Gamma^4_{\mu\lambda} = \frac{1}{2}\hat{G}^{44}\left(\partial_\lambda A_\mu+\partial_\mu A_\lambda\right) + \frac{1}{2}\hat{G}^{4\nu}\left(\partial_\lambda\hat{G}_{\mu\nu}+\partial_\mu\hat{G}_{\lambda\nu}-\partial_\nu\hat{G}_{\mu\lambda}\right)$$

$$= \frac{1}{2}(1+\cancel{A^2})(\partial_\mu A_\lambda+\partial_\lambda A_\mu) - \frac{1}{2}A^\nu\hat{\Gamma}^\rho_{\mu\lambda} - \frac{1}{2}A^\nu\underbrace{\left\{\partial_\lambda(A_\mu A_\nu)+\partial_\mu(A_\lambda A_\nu)-\partial_\nu(A_\mu A_\lambda)\right\}}_{\cancel{A_\nu(\partial_\lambda A_\mu+\partial_\mu A_\lambda)}+A_\mu F_{\lambda\nu}+A_\lambda F_{\mu\nu}},$$

$$= -\frac{1}{2}A_\rho\hat{\Gamma}^\rho_{\mu\lambda} + \frac{1}{2}(\partial_\mu A_\lambda+\partial_\lambda A_\mu) - \frac{1}{2}A^\nu\left(A_\lambda F_{\mu\nu}+A_\mu F_{\lambda\nu}\right)$$

$$\Gamma^N_{N4} = \Gamma^\mu_{\mu 4} + \cancel{\Gamma^4_{44}} = \frac{1}{2}\, \delta^\mu_\sigma \hat{g}^{\sigma\nu} F_{\mu\nu} = 0$$

$$\Gamma^N_{N\sigma} = \Gamma^\mu_{\mu\sigma} + \Gamma^4_{4\sigma} = \hat{\Gamma}^\nu_{\nu\sigma} + \cancel{\frac{1}{2} A^\nu F_{\sigma\nu}} - \cancel{\frac{1}{2} A^\nu F_{\sigma\nu}} = \hat{\Gamma}^\nu_{\nu\sigma}$$

$$R = \hat{G}^{M\rho} \partial_\rho \Gamma^N_{MN} - \hat{G}^{MR} \partial_\nu \Gamma^\nu_{MR} + \hat{G}^{MR} \left(\Gamma^\Sigma_{MN} \Gamma^N_{R\Sigma} - \Gamma^\Sigma_{MR} \Gamma^N_{N\Sigma} \right)$$

$$= \hat{g}^{\mu\rho} \partial_\rho \hat{\Gamma}^\nu_{\mu\nu} - \hat{g}^{\mu\rho} \partial_\nu \Gamma^\nu_{\mu\rho} - 2 \hat{G}^{\mu 4} \partial_\nu \Gamma^\nu_{\mu 4} - \hat{G}^{MR} \Gamma^\sigma_{MR} \Gamma^N_{N\sigma} + \underbrace{\hat{G}^{MR} \Gamma^\Sigma_{MN} \Gamma^N_{R\Sigma}}$$

$$= \hat{R} - \hat{g}^{\mu\lambda} \partial_\sigma \left\{ \hat{g}^{\sigma\nu} A_\lambda F_{\mu\nu} \right\} + A^\mu \partial_\sigma \left(\hat{g}^{\sigma\nu} F_{\mu\nu} \right) - 2 \hat{G}^{\mu 4} \Gamma^\sigma_{\mu 4} \hat{\Gamma}^\nu_{\nu\sigma} - \hat{g}^{\mu\rho} \delta\Gamma^\sigma_{\mu\rho} \hat{\Gamma}^\nu_{\nu\sigma} + \dots$$

$$= \hat{R} - \hat{g}^{\mu\lambda} \hat{g}^{\sigma\nu} (\partial_\sigma A_\lambda) F_{\mu\nu} + \cancel{A^\mu \hat{g}^{\sigma\lambda} F_{\mu\lambda} \hat{\Gamma}^\nu_{\nu\sigma}} - \cancel{\hat{g}^{\mu\lambda} \hat{g}^{\sigma\rho} A_\lambda F_{\mu\rho} \hat{\Gamma}^\nu_{\nu\sigma}} + \dots$$

$$= \hat{R} + \frac{1}{2} F^2 + \left\{ \hat{G}^{MR} \underbrace{\left\{ \Gamma^\sigma_{MN} \Gamma^N_{R\sigma} + \Gamma^4_{MN} \Gamma^N_{R4} \right\}}_{\Gamma^\sigma_{M\nu} \Gamma^\nu_{R\sigma} + 2 \Gamma^\sigma_{M4} \Gamma^4_{R\sigma} + \Gamma^4_{M4} \Gamma^4_{R4}} \right.$$

$$\{\ \} = \hat{G}^{44} \Gamma^\sigma_{4\nu} \Gamma^\nu_{4\sigma} + \hat{G}^{4\mu} \left(2 \Gamma^\sigma_{\mu\nu} \Gamma^\nu_{4\sigma} + 2 \Gamma^\sigma_{\mu 4} \Gamma^4_{4\sigma} \right) + \left(\Gamma^\sigma_{\mu\nu} \Gamma^\nu_{\rho\sigma} + 2 \Gamma^\sigma_{\mu 4} \Gamma^4_{\rho\sigma} + \Gamma^4_{\mu 4} \Gamma^4_{\rho 4} \right) \hat{g}^{\mu\rho}$$

$$= (1+A^2) \frac{1}{4} \hat{g}^{\sigma\lambda} F_{\nu\lambda} \hat{g}^{\nu\rho} F_{\sigma\rho} - A^\mu \left(\hat{g}^{\nu\lambda} F_{\sigma\lambda} \Gamma^\sigma_{\mu\nu} - \frac{1}{2} \hat{g}^{\sigma\lambda} F_{\mu\lambda} A^\rho F_{\sigma\rho} \right) + \Gamma^\sigma_{\mu\nu} \Gamma^\nu_{\rho\sigma} \hat{g}^{\mu\rho}$$
$$+ \cancel{\hat{g}^{\mu\rho} \hat{g}^{\sigma\nu} F_{\mu\nu} \Gamma^4_{\rho\sigma}} + \frac{1}{4} A^\nu F_{\mu\nu} A^\lambda F_{\rho\lambda} \hat{g}^{\mu\rho}$$

$$= -(1+A^2) \frac{1}{4} F^2 - \frac{1}{4} \left(A^\nu F_{\mu\nu} \right)^2 - \cancel{A^\mu \hat{\Gamma}^\sigma_{\mu\nu} \hat{g}^{\nu\lambda} F_{\sigma\lambda}} - \frac{1}{2} \hat{g}^{\nu\lambda} F_{\sigma\lambda} A^\mu \hat{g}^{\sigma\rho} \left(A_\mu F_{\nu\rho} + A_\nu F_{\mu\rho} \right)$$
$$+ \hat{g}^{\mu\rho} \hat{\Gamma}^\nu_{\rho\sigma} \hat{g}^{\sigma\lambda} \cancel{\left(A_\mu F_{\nu\lambda} + A_\nu F_{\mu\lambda} \right)} + \frac{1}{4} \hat{g}^{\mu\rho} \hat{g}^{\sigma\lambda} \left(A_\mu F_{\nu\lambda} + A_\nu F_{\mu\lambda} \right) \hat{g}^{\nu\tau} \times \left(A_\rho F_{\sigma\tau} + A_\sigma F_{\rho\tau} \right)$$

$$= -\frac{1}{4} (1 + \cancel{A^2}) F^2 - \frac{1}{4} \cancel{\left(A^\nu F_{\mu\nu} \right)^2} + \frac{1}{2} \cancel{A^2} F^2 + \frac{1}{2} \cancel{\left(A^\nu F_{\mu\nu} \right)^2} - \frac{1}{4} \cancel{A^2} F^2 - \frac{1}{2} \cancel{\left(A^\nu F_{\mu\nu} \right)^2} + \frac{1}{4} \cancel{\left(A^\nu F_{\mu\nu} \right)^2}$$

$$= -\frac{1}{4} F^2$$

$$\Rightarrow \hat{R}^{(5)} = \hat{R}^{(4)} + \frac{1}{4} F^2$$

$$\sqrt{g}\, R^{(5)}\left[e^{2\phi}\hat{G}_{\mu\nu}\right] \underset{4d}{=} e^{3\phi}\sqrt{\hat{G}}\left[\hat{R}^{(4)} + \frac{1}{4}F^2 - 8(\hat{D}\phi)^2 - 12(\hat{D}\phi)^2\right] \qquad (*)$$

$$\Downarrow \; 24(\hat{D}\phi)^2$$

$$\underset{4d}{=} e^{3\phi}\sqrt{\hat{G}}\left[\hat{R} + \frac{1}{4}F^2 + 12(\hat{D}\phi)^2\right]$$

$$= 12e^{2\omega}(D\phi)^2$$

$$\hat{G}_{\mu\nu} = e^{2\omega} g_{\mu\nu} \Rightarrow \qquad e^{2\omega}\left[R - 6D^2\omega - 6(D\omega)^2\right]$$

$$\omega = -\frac{3}{2}\phi \Rightarrow \qquad = R - 6\cdot\frac{9}{4}(D\phi)^2 + 12(D\phi)^2 + \frac{1}{4}e^{3\phi}F^2$$

$$12 - \frac{27}{2} = -\frac{3}{2} \qquad = R - \frac{3}{2}(\partial\phi)^2 + \frac{1}{4}e^{3\phi}F^2$$

$$\phi = \frac{1}{\sqrt{3}}\varphi \qquad = R - \frac{1}{2}(\partial\varphi)^2 + \frac{1}{4}e^{\sqrt{3}\varphi}F^2$$

$$G_{MN} = e^{2/\sqrt{3}\,\varphi}\begin{pmatrix} e^{-\sqrt{3}\varphi} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2\kappa_5^2}\int dy\, R^{(5)} = \frac{2\pi R}{2\kappa_5^2}\left\{R^{(4)} - \frac{1}{2}(\partial\varphi)^2 + \frac{1}{4}e^{\sqrt{3}\varphi}F^2 + \sum_{n\neq 0}\left(\text{massive spin-2 fields}^{KK}\right)\right.$$

Kaluza-Klein idea: unify gravity with electromagnetism in 5d

φ: "Radion" excitation

Radion: • $G_{44} = e^{2\phi}$ $\qquad (*) \Rightarrow \langle e^{3\phi}\rangle$ changes $2\pi R$

$\Rightarrow \langle e^{\sqrt{3}\varphi}\rangle$ fixes the radius of the extra dimension

EFT: integrate out massive KK modes $\Rightarrow$ EFT for light modes

$\Rightarrow$ higher dim eff operators

4d $\leftrightarrow$ 5d couplings $\qquad$ 1) U(1): $\displaystyle \frac{1}{g_4^2} = \frac{2\pi R}{g_5^2}$ $\qquad [g_5^2] =$ length

2) gravity: $\displaystyle M_p^2 = \frac{1}{\kappa_4^2} = \frac{2\pi R}{\kappa_5^2} = 2\pi R\, M_5^3$

– Forces in extra dims

1) Coulomb potential

$$\vec{B}=0 \quad \rho = Q\,\delta^3(x) \qquad \vec{\nabla}\vec{E}=\rho \quad \vec{\nabla}\times\vec{E}=0 \quad \Rightarrow \quad \vec{E}=\vec{\nabla}\phi \,, \quad \vec{\nabla}^2\Phi=\rho$$

$$\Phi \sim Q\int d^3k\,\frac{e^{i\vec{k}\vec{x}}}{k^2} \sim \frac{1}{r} \qquad r=|\vec{x}| \qquad \text{(by dim analysis)}$$

2) Newtonian potential : same with $Q \to GM^2$

n extra dim : $\int d^3k \to \int d^{3+n}k \quad \Rightarrow \quad \frac{1}{r} \to \frac{1}{r^{1+n}}$

3d → (3+n)d due to KK excitations

massive particle:

$$\int d^3k\,\frac{e^{i\vec{k}\vec{x}}}{k^2+m^2} \sim \int dk\,k^2\int_{-1}^{1} d\cos\theta\,\frac{e^{ikr\cos\theta}}{k^2+m^2} \sim \frac{1}{r}\underbrace{\int_0^\infty dk\,\frac{k\sin kr}{k^2+m^2}}_{\sim\, e^{-mr}} \qquad m>0$$

$\Rightarrow$ Yukawa potential : $\dfrac{e^{-mr}}{r}$ $\qquad m = 0,\ \underbrace{\frac{1}{R},\ \frac{2}{R},\ \ldots}_{\text{multiplicity } 2}$

$$\Rightarrow \frac{1}{r}\left\{1+2e^{-r/R}+2e^{-2r/R}+\ldots\right\} \equiv V$$

$R\to 0 \Rightarrow V \to \frac{1}{r}$ (3d)

R finite $\Rightarrow$ $V \approx \frac{1}{r}\left\{1+2\left(\frac{1}{1-e^{-r/R}}-1\right)\right\} = \frac{1}{r}\left\{1+2\,\frac{e^{-r/R}}{1-e^{-r/R}}\right\}$

$R\to\infty \Rightarrow V \sim \dfrac{2R}{r^2} \qquad Q \to Q_5 = 2R\,Q_4$

Generalizations, orbifolds, symmetry breaking by compactification

- Generalizations

. Non-abelian vectors $A^a_M \leftarrow$ gauge group

$$A^a_M = \begin{pmatrix} A^a_\mu \\ \phi^a \equiv A^a_4 \end{pmatrix} \Rightarrow \operatorname{Tr} F^2_{MN}$$

adjoint scalar

$$\int dy \operatorname{Tr} F^2_{MN} = \operatorname{Tr} F^2_{\mu\nu}(x) + \operatorname{Tr}(\partial\phi)^2 + \sum_{n\neq 0} \Big(\text{massive vectors} \Big)$$

$$\text{massive vectors} = \operatorname{Tr}\left(F^n_{\mu\nu}\right)^2$$

$$F^n_{\mu\nu} = \partial_\mu A^n_\nu - \partial_\nu A^n_\mu + \sum_m \left[A^m_\mu , A^{n-m}_\nu \right]$$

- More extra dims $n > 1$ KK excitations (besides the 0-modes)

$$\cdot\ A_M \Rightarrow \begin{pmatrix} A_\mu \\ \phi^i \end{pmatrix} \begin{matrix} \rightarrow \text{massive vectors} \\ \rightarrow n-1 \text{ massive scalars (adjoints)} \end{matrix}$$

$$\cdot\ G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\mu i} \\ G_{\nu j} & G_{ij} \end{pmatrix}$$

0-modes: 4d graviton, n vectors, $\frac{n(n+1)}{2}$ scalars

KK-modes: spin-2 $G^n_{\mu\nu}$, $n-1$ vectors, $\frac{n(n-1)}{2}$ scalars

$$\frac{n(n-1)}{2} = \frac{n(n+1)}{2} - 1 - (n-1)$$

massive degrees of freedom: $5 + 3(n-1) + \frac{n(n-1)}{2} = 3n + 2 + \frac{n(n-1)}{2} = \frac{n^2+5n+4}{2}$

$$= \frac{(n+4)(n+1)}{2} = \frac{D(D-3)}{2}$$

$\uparrow$ graviton in D dim

$$\frac{D(D+1)}{2} - \underset{\substack{\uparrow \\ \text{gauge} \\ \text{cond}}}{D} - \underset{\substack{\uparrow \\ \text{Gauss law}}}{D} = \frac{D^2-3D}{2}$$

Normalized fields: $\tilde{\Phi}_{\vec{n}} = \Phi_{\vec{n}} / \sqrt{V_{int}} \leftarrow$ internal volume

• Scalars G_{ij} : non-trivial compactification space
sizes and (complex structure) shape of the manifold

E.g. $n=2$ $\begin{pmatrix} G_{44} & G_{45} \\ G_{45} & G_{55} \end{pmatrix} \Rightarrow$ $\sqrt{G}$: radius
G_{45}/G_{55} , G_{44}/G_{55} : complex structure/shape

– Chirality and orbifolds

4d chirality from extra dims : non trivial

Higher dim spinor $\Rightarrow$ L,R symmetric in 4d

E.g. $D=5 \Rightarrow$ γ-matrices : $\{\gamma^\mu, \gamma^5\} = \gamma^M$

$\Rightarrow$ 5d spinor $\equiv$ 4d Dirac spinor (L + R component)

Need non-trivial manifold

Orbifolds : special limits allowing chirality
at the expense of singular (fixed) points

Ex: one dim on an interval $\equiv S^1/\mathbb{Z}_2$ ← $y \equiv y + 2\pi R$; $y \to -y$

fixed points under the $\mathbb{Z}_2$

Boundary conditions

- $\Phi(y+2\pi R) \equiv \Phi(y) \Rightarrow \Phi(-y) = \pm\,\phi(y)$ — $+$: even (e), $-$: odd (o)

$$\left.\begin{array}{l} \phi_{even} = \sum_n \cos\frac{n}{R}y\ \phi_n \quad \frac{|n\rangle+|-n\rangle}{\sqrt{2}}\ KK \\ \phi_{odd} = \sum_n \sin\frac{ny}{R}\ \phi_n \quad \frac{|n\rangle-|-n\rangle}{\sqrt{2}}\ KK \end{array}\right\}$$ invariant under $y\to -y$ and $y \to y+2\pi R$

ϕ_{odd}: no zero-modes $\quad \phi_{odd}(0) = \phi_o(\pi R) = 0 \quad , \phi_o = 0 \quad n: odd$

ϕ_{even}: $\quad n$: even $\quad \phi_{even}(0) = \phi_o = -\phi_{even}(\pi R)$

- Fermions

$$\psi(-y) = \gamma^4\psi(y) \equiv \gamma^5\psi(y) \Rightarrow \psi_L = \frac{1+\gamma_5}{2}\psi \ \text{invariant}$$
$$\psi_R = \frac{1-\gamma_5}{2}\psi \to -\psi_R$$

$\Rightarrow$ 0-modes only for $\psi_L \Rightarrow$ 4d chirality

- Gauge fields

$$A_M(-y) = \begin{pmatrix} A_\mu(y) \\ -A_4(y) \end{pmatrix} \Rightarrow \begin{array}{l} A_\mu: \text{even} \\ A_4: \text{odd} \end{array}$$

$\Rightarrow A_\mu^{(n)}$ n even massive + 0-mode , no scalar

- metric

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & \overbrace{G_{\mu 4}}^{odd} \\ \underbrace{G_{4\nu}}_{odd} & \underbrace{G_{44}}_{even} \end{pmatrix} \Rightarrow G_{\mu\nu}^{(n)} \quad n \text{ even}$$

G_{44} = radion

no vectors

- fixed points : boundary states (twisted)

fixed points : singular (breakdown of EoM)

$\Rightarrow$ in general extra states "localized" at the fixed points (boundaries)
4d anomalies

model dependent (twisted in string theory)

$\Rightarrow$ notion of "branes"

- Coordinate dependent compactification, mass generation and symmetry breaking

- Scherk-Schwarz 79

1) symmetry of higher dim theory $\Rightarrow$ general boundary conditions

periodic up to symmetry transformation

e.g. $U(1)$ phase : $\Phi(y+2\pi R) = e^{2i\pi\omega}\,\Phi(y)$ ← complex field

$$\Rightarrow \Phi(y) = \sum_n e^{iy\frac{n+\omega}{R}}\,\Phi_n$$

$$S^1/\mathbb{Z}_2 \Rightarrow \quad \Phi_+ = \sum_n \cos y\frac{n+\omega}{R}\,\Phi_n^+ \qquad \Phi_- = \sum_n \sin y\frac{n+\omega}{R}\,\Phi_n^-$$

$\Rightarrow$ 0-mode gets a mass ω/R , $|n\rangle, |-n\rangle$ split (for an orbifold)

2) gauge symmetry breaking e.g. $SU(2) \to U(1)$ (T_3 generator)

$$\Phi(y+2\pi R) = e^{-2i\pi\omega t_3}\,\Phi(y)\,e^{2i\pi\omega t_3} \qquad \Phi \equiv A_M$$

$\{\sigma^\pm, t_3\} = 0 \quad \Rightarrow \quad t_3 = \frac{\sigma_3}{2}$

$\Phi_\pm(y+2\pi R) = e^{-4i\pi\omega t_3}\, \Phi_\pm(y) \quad , \quad \phi_3(y+2\pi R) = \phi_3(y)$

$= \left[\cos 2\pi\omega - i\sigma_3 \sin 2\pi\omega\right] \phi_\pm(y)$

$\sigma^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2) = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \pm \frac{i}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}$

$\Rightarrow \sigma^+ = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix} \quad \sigma^3\sigma^+ = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & 1 \\ 0 & \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma^+$

$\sigma^3\sigma^- = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & 0 \\ 1 & \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -\sigma^-$

$\Rightarrow \phi_\pm(y+2\pi R) = e^{\mp 2i\pi\omega}\, \phi_\pm(y)$

$\Rightarrow \phi^\pm_n$ gets a mass^2 $(n \pm \omega)^2$

$\Rightarrow A^\pm_{\mu,n}$ absorbs $A^\pm_{4,n}$, $A^3_{\mu,0}$ massless , $A^3_{4,0}$ massless

$SU(2) \to U(1)$ by a vev of $\langle A_4 \rangle \sim \omega\sigma^3$

3) Supersymmetry breaking

use for $U(1)$ an R-symmetry

fermions (gravitino, gaugino) transform $\Rightarrow$ SUSY is broken

frequencies shifted by ω: $n+\omega \equiv m_n$

usually R-symmetry discrete $\Rightarrow$ ω is quantized

special case $\omega = \frac{1}{2}$ $\Rightarrow$ fermions antiperiodic $\Rightarrow$ $\frac{1}{2}$-integer frequencies

as in finite temperature $y \to$ time

Branes in extra dimensions

- Brane : hypersurface of p-spatial dimensions (p-brane)

  embedded in a higher dimensional bulk ($D = 4+n > p+1$)

Field theory: topological defect

String theory: D-branes defined by the ends of open strings

branes can carry "charge" (necessary for stability)

point particle ≡ 0-brane can be electric source for gauge field (1-form)

$$e_0 \int dx^\mu A_\mu$$

string ≡ 1-brane : for 2-form gauge potential (2-index tensor)

$$e_1 \int dx^\mu \wedge dx^\nu A_{\mu\nu}$$

⋮

p-brane ⟶ (p+1)-form gauge potential

$$e_p \int \underbrace{dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}}} A^{(p+1)}_{\mu_1 \cdots \mu_p}$$

$$\int d^{p+1}\xi \ \varepsilon^{\alpha_1 \cdots \alpha_n} \partial_{\alpha_1} X^{\mu_1} \cdots \partial_{\alpha_{p+1}} X^{\mu_{p+1}} A^{(p+1)}_{\mu_1 \cdots \mu_{p+1}} \quad (*)$$

ξ^α ⟶ brane world volume coordinates

⟶ after defined branes

new notation

$$\int d^{p+1}x \ \varepsilon^{\alpha_1 \cdots \alpha_{p+1}} \partial_{\alpha_1} Y^{M_1} \cdots \partial_{\alpha_{p+1}} Y^{M_{p+1}} A^{(p+1)}_{M_1 \cdots M_{p+1}}$$

eg. D3-brane in 10 d: $ds^2 = f^{-1/2} dx_{\parallel}^2 + f^{1/2} (dr^2 + r^2 d\Omega_5^2)$ ⟶ unit 5-sphere

$$f = 1 + \frac{4\pi g N l^4}{r^4} \ : \text{harmonic function} \ \Delta_6 \frac{1}{r^4} = \delta^6(y)$$

$3+3$ $1+3$

two branes: gravity attractive force
A repulsive " (due e.m.) } ⇒ they cancel

Brane analogy with an extremal charged black hole

Charged BH (Reissner-Nordström solution)

$$ds^2 = -\left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)dt^2 + \left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_2$$

2-sphere of unit radius ($d\theta^2 + \sin^2\theta\, d\varphi^2$)

$r_s = 2GM$

$r_Q^2 = e^2 G$

$\Rightarrow$ two horizons at $r = r_\pm = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right)$

$A_\mu = \left(-\frac{e}{r}, \vec{0}\right)$

extremal case: $r_+ = r_- \Rightarrow r_s = 2r_Q \Rightarrow \boxed{GM^2 = e^2}$

$$\Rightarrow ds^2 = -\left(1-\frac{r_Q}{r}\right)^2 dt^2 + \underbrace{\left(1-\frac{r_Q}{r}\right)^{-2}dr^2 + r^2 d\Omega_2}_{f^2(R)\left[dR^2 + R^2 d\Omega_2\right]}$$

$$r = Rf(R) \Rightarrow \frac{dr}{1-r_Q/r} = f(R)\,dR = \frac{r}{R}dR \Rightarrow \frac{dr}{r-r_Q} = \frac{dR}{R}$$

$$\Rightarrow r = r_Q + R \quad , \quad f(R) = \frac{r}{R} = 1 + \frac{r_Q}{R} \quad , \quad 1-\frac{r_Q}{r} = \frac{R}{r} = \frac{1}{f(R)}$$

$$\Rightarrow ds^2 = -\frac{1}{f^2(R)}dt^2 + f^2(R)\left(dR^2 + R^2 d\Omega_2\right)$$

with $f(R) = 1 + \frac{r_Q}{R}$: harmonic function

$$\Delta_3 f(R) = r_Q\, \delta^3(x)$$

$\Rightarrow$ extremal BH $\equiv$ 0-brane

↳ point particle with charge $e = \sqrt{GM^2}$

force between two:

$$\frac{GM^2}{r} - \frac{e^2}{r} = 0 \qquad \text{for } e^2 = \sqrt{GM^2}$$

gravity attractive — em - repulsive

no force $\Rightarrow$ stability

- branes with charge in the extremal limit:
  generalization of this situation

* Ex: 3-brane world

X^M ; $M=0,1,\dots,3+n$: spacetime coordinates of the bulk

x^μ ; $\mu=0,1,2,3$ coordinates of the 3-brane (our spacetime)

$Y^M(x)$: positions of the 3-brane in the extra dims

$(4+n)$-dim metric $G_{MN}(X)$

Bulk fields : $B(X)$

Brane-localized fields : $\Phi(x)$

Vacuum state: $G_{MN}=\eta_{MN}$ (flat space bulk)

$$Y^M=\delta^M_\mu x^\mu$$

→ describe fluctuations around the vacuum state

$$S_{bulk}=-\int d^{4+n}X\ \sqrt{|G|}\ \left\{ M_*^{2+n}\ R^{(4+n)}+\Lambda+\text{other bulk fields}\right\}$$

Induced metric on the brane:

$$ds^2=G_{MN}\,dY^M(x)\,dY^N(x)=\underbrace{G_{MN}\frac{\partial Y^M}{\partial x^\mu}\frac{\partial Y^N}{\partial x^\nu}}_{g_{\mu\nu}\ :\ \text{induced metric}}dx^\mu dx^\nu$$

$G_{MN}(Y(x))$

Symmetries of the Eff Action : general parametrization of the bulk
+ " " of the brane coordinates

⇒ ex. $\partial_\mu Y^M$: vector under bulk coordinate transformations
" " brane

$g_{\mu\nu}$: tensor under brane
scalar " bulk transfs

etc.

$$S_{brane} = -\int d^4x \sqrt{|g|} \left\{ T + R^{(4)} + \frac{1}{2} g^{\mu\nu} D_\mu \Phi D_\nu \Phi + \frac{1}{4} F^2_{\mu\nu} + \dots \right\}$$

gauge choice of brane coord transf. $x^\mu \to x'^\mu(x)$

4 conditions $\Rightarrow$ $Y^\mu(x) = x^\mu$, $Y^m(x)$ $m = 4, 5, \dots, 3+n$

physical degrees of freedom

brane positions in the n-extra dims

$G_{MN} = \eta_{MN} + \frac{1}{M_*^{1+n/2}} h_{MN}$ ← higher dim graviton

Brane fluctuations: $g_{\mu\nu} = G_{MN} \partial_\mu Y^N \partial_\nu Y^N = \eta_{\mu\nu} + \partial_\mu Y^m \partial_\nu Y_m + \dots$

$$\Rightarrow \det g = -1 - \partial_\mu Y^m \partial^\mu Y_m + \dots$$

T: brane tension ($T^{1/4} \ll M_*$ to neglect back reaction)

canonically normalized field: $Z^m = T^{1/2} Y^m$

For $Y = 0$ $g_{\mu\nu} = G_{\mu\nu}(x^\lambda, x^m = 0)$

← brane position

$\Rightarrow$ For SM localized on the 3-brane $S_{SM} = \int d^4x \, \mathcal{L}_{SM} \sqrt{g} (g_{\mu\nu}, \Phi, \Psi, \dots)$

$\Rightarrow$ Coupling of the brane fields to graviton:

$$S_{int} = \int d^4x \, T^{\mu\nu}_{SM} \frac{h_{\mu\nu}(x_\mu)}{M_*^{n/2+1}}$$

KK excitations: $h_{\mu\nu}(x,y) = \sum_n e^{iny/R} \frac{h^{(n)}_{\mu\nu}}{\sqrt{2\pi R}}$

e.g. for 1 extra dim

↗

in general $n \to \vec{n}$

$R \to \vec{R}$ for toroidal (or more general)

$\sqrt{2\pi R} \to \sqrt{V_{int}}$

$$\Rightarrow S_{int} = \int d^4x \; T^{\mu\nu}_{SM} \sum_k \frac{h^{k}_{\mu\nu}}{\sqrt{V_n}} \frac{1}{M_*^{n/2+1}}$$

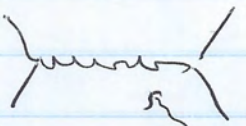
int volume

4d M_p $\qquad (M_p^2 = M_*^{2+n} V_n)$

$\Rightarrow$ each KK mode couples with the same strength M_p (as the 0-mode)

$\longrightarrow$ brane charge from p. 16

- Exchange of KK modes between brane sources



Ex of bulk gauge field or bulk graviton

propagator: $\dfrac{1}{k^2 + m_0^2 + \frac{n^2}{R^2}} \times$ tensor structure $\qquad$ (one extra dim, m_0: 5d mass, k: 4d momentum)

$$\sum_{n\in\mathbb{Z}} \frac{1}{a^2+n^2} = \frac{\pi}{a}\coth \pi a \Rightarrow$$

$$\sum_{\substack{KK\text{-modes}\\ \neq 0}} = \left(\frac{\pi}{Ra}\coth \pi R a - \frac{1}{R^2a^2}\right) R^2 g^2 \qquad a = \sqrt{k^2+m_0^2}$$

g: 4d coupling

$$\Sigma = g^2\left(\frac{\pi R}{a}\coth \pi R a - \frac{1}{a^2}\right)$$

(can be $\frac{1}{M_p}$ for gravity)

$$\coth x = \begin{cases} \pm 1 & x \to \pm\infty \\ \dfrac{1}{x} + \displaystyle\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} x^{2n-1}}{(2n)!} = \frac{1}{x} + \frac{x}{3} + \dots & x \to 0 \end{cases}$$

Bernoulli numbers

limits:

- 4d limit $R \to 0 \Rightarrow \Sigma \to 0$
- $k = 0$ (0-momentum) $\Rightarrow a = m_0$

  $m_0 = (k=)0 \Rightarrow \Sigma \to \frac{g^2}{3} \pi^2 R^2$: low energy eff operator of the form $(\text{current})^2$

  $T_{\mu\nu}$ for gravity

- $m_0 = 0 \Rightarrow a = k \left(= \sqrt{k^2}\right)$
- $Rk \to \infty \Rightarrow \Sigma \sim g^2 \frac{\pi R}{k} \sim \frac{g_5^2}{2k}$ where $g^2 = \frac{g_5^2}{2\pi R}$

  e.g. $\frac{1}{g_5^2} F^2 \to \frac{2\pi R}{g_5^2} \equiv \frac{1}{g_4^2}$ or $M_p^2 = M_*^3 \, 2\pi R$

  g_4^2 $\quad$ g_5^2 for gravity

  more dims: $2\pi R \to V_n$: volume of n-extra dims

- More than 1 extra dims $n \geq 2$: sum over $\vec{n}$ diverges

  for $n=2$ logarithmic ; for $n \geq 3$ power Λ^{n-2}

  $\Rightarrow$ need regularization: width δ: $e^{-\delta \vec{n}^2}$