Representation of the Lorentz Boosts in 6-dimensional Space-time

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1 Introduction

When we consider parallel transport of a 4-vector of velocity, the displaced 4-vector is again a 4-vector of velocity. But if we consider the 4-vector of velocity as a Lorentz boost, then its parallel displacement may not be a boost, because may contain a space rotation, and can simultaneously give information for both the velocity and space rotation of the considered body. This is the main motivation for the present paper, where we research a model of 3-dimensional time. The evidence of the 3-dimensional time appears also in the quantum mechanics, where besides the three spatial coordinate operators appear 3 impulse coordinates, which are indeed temporal coordinates.

Albert Einstein and Henri Poincare many years ago thought about 3-dimensional time, such that the space and time would be of the same dimension. At present time some of the authors [1-5,7-9] propose multidimensional time in order to give better explanation of the quantum mechanics and the spin.

2 Basic results

At each moment the set of all moving frames can be considered as a principal bundle over $\mathbb{R}^3$, where the structural group is the Lorentz group of transformations. This bundle will be called *space-time bundle*. This bundle can be parameterized by the following 9 coordinates

$$\{x, y, z\}, \quad \{x_s, y_s, z_s\}, \quad \{x_t, y_t, z_t\},$$

such that the first 6 coordinates parameterize the subbundle with the fiber $SO(3, \mathbb{R})$. So this approach in the SR will be called 3+3+3-dimensional model. Indeed, to each body are related 3 coordinates for the position, 3 coordinates for the space rotation and 3 coordinates to its velocity.

Firstly, we consider the analog of the Lorentz boosts from the 3+1-dimensional space-time. The coordinates x_s, y_s, z_s and x_t, y_t, z_t are functions of basic space coordinate x, y, and z, and assume that the Jacobi matrices

$$V = \begin{bmatrix} \frac{\partial x_s}{\partial x} & \frac{\partial x_s}{\partial y} & \frac{\partial x_s}{\partial z} \\ \frac{\partial y_s}{\partial x} & \frac{\partial y_s}{\partial y} & \frac{\partial y_s}{\partial z} \\ \frac{\partial z_s}{\partial x} & \frac{\partial z_s}{\partial y} & \frac{\partial z_s}{\partial z} \end{bmatrix} \quad \text{and} \quad V^* = \begin{bmatrix} \frac{\partial x_t}{\partial x} & \frac{\partial x_t}{\partial y} & \frac{\partial x_t}{\partial z} \\ \frac{\partial y_t}{\partial x} & \frac{\partial y_t}{\partial y} & \frac{\partial y_t}{\partial z} \\ \frac{\partial z_t}{\partial x} & \frac{\partial z_t}{\partial y} & \frac{\partial z_t}{\partial z} \end{bmatrix} \tag{2.1}$$

are respectively symmetric and antisymmetric. Further, let us denote $X = x_s + ix_t$, $Y = y_s + iy_t$, $Z = z_s + iz_t$, such that the Jacobi matrix $\mathcal{V} = [\frac{\partial(X,Y,Z)}{\partial(x,y,z)}]$ is Hermitian and $\mathcal{V} = V + iV^*$.

The antisymmetric matrix V^* depends on 3 variables and its general form can be written as

$$V^* = \frac{-1}{c\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 0 & v_z & -v_y \\ -v_z & 0 & v_x \\ v_y & -v_x & 0 \end{bmatrix}. \tag{2.2}$$

From (2.2) we can join to V^* a 3-vector $\vec{v} = (v_x, v_y, v_z)$, which transforms as a 3-vector. Namely, let we choose an orthogonal 3×3 matrix P, which determines a space rotation on the base $B = \mathbb{R}^3$, applying to the coordinates x, y, z. Then this transformation should also be applied to both sets of coordinates $\{x_s, y_s, z_s\}$ and $\{x_t, y_t, z_t\}$. Hence the matrix V^* maps into $PV^*P^{-1} = PV^*P^T$, which corresponds to the 3-vector $P\cdot\vec{v}$. Thus $\vec{v}\mapsto P\cdot\vec{v}$, and $\vec{v}$ is a 3-vector.

It is natural to assume that $\mathcal{V}$ should be presented in the form

$$\mathcal{V} = e^{iA} = \cos A + i\sin A.$$

Assume that A is an antisymmetric real matrix, which is given by

$$A = \begin{bmatrix} 0 & -k\cos\gamma & k\cos\beta \\ k\cos\gamma & 0 & -k\cos\alpha \\ -k\cos\beta & k\cos\alpha & 0 \end{bmatrix},$$

where $\vec{v} = c(\cos\alpha, \cos\beta, \cos\gamma)\tanh(k)$ and $(\cos\alpha, \cos\beta, \cos\gamma)$ is a unit vector of the velocity vector. As a consequence we obtain

$$\sin A = \frac{-1}{c\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 0 & v_z & -v_y \\ -v_z & 0 & v_x \\ v_y & -v_x & 0 \end{bmatrix}, \tag{2.3}$$

i.e. that $V^* = \sin A$ is given by (2.2), while the symmetric 3×3 matrix $\cos A$ is given by

$$(\cos A)_{ij} = V_4\delta_{ij} + \frac{1}{1+V_4}V_iV_j, \tag{2.4}$$

where $(V_1, V_2, V_3, V_4) = \frac{1}{ic\sqrt{1-\frac{v^2}{c^2}}}(v_x, v_y, v_z, ic)$.

From (2.1) and (2.2) the time vector in this special case is given by

$$(x_t, y_t, z_t) = \frac{\vec{v}}{c\sqrt{1-\frac{v^2}{c^2}}} \times (x, y, z) + (x_t^0, y_t^0, z_t^0), \qquad (2.5)$$

where (x_t^0, y_t^0, z_t^0) does not depend on the basic coordinates. The coordinates x_t, y_t, z_t are independent and they cover the Euclidean space $\mathbb{R}^3$ or an open subset of it. But the Jacobi matrix $[\frac{\partial(x_t,y_t,z_t)}{\partial(x,y,z)}]$ is a singular matrix as antisymmetric matrix of order 3, where the 3-vector of velocity maps into zero vector. So the quantity $(x_t, y_t, z_t) \cdot \vec{v}$ does not depend on the basic coordinates and hence we assume that **it determines the 1-dimensional time t measured from the basic coordinates**. For example, if velocity is parallel to the z-axis, then z_t does not depend on the basic coordinates because $\frac{\partial z_t}{\partial x} = \frac{\partial z_t}{\partial y} = \frac{\partial z_t}{\partial z} = 0$ and hence z_t is proportional with the time from the basic coordinate system. Further, one can easily verify that $(1-\frac{v^2}{c^2})^{-1/2}(\vec{v} \times (x, y, z)) = (1-\frac{v^2}{c^2})^{-1/2}(\vec{v} \times (\cos A)^{-1}(x_s, y_s, z_s)) = \vec{v} \times (x_s, y_s, z_s)$ for simultaneous points in basic coordinates. So (2.5) becomes

$$(x_t, y_t, z_t) = \frac{\vec{v}}{c} \times (x_s, y_s, z_s) + \vec{c} \cdot \Delta t, \qquad (2.6)$$

where $\vec{c} = \frac{\vec{v}}{v} \cdot c$.

3 Local isomorphism between $O_+^{\uparrow}(1,3)$ and $SO(3,\mathbb{C})$

Let us consider the following mapping $F : O_+^{\uparrow}(1,3) \to SO(3,\mathbb{C})$ given by

$$\begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1-\frac{1}{1+V_4}V_1^2 & -\frac{1}{1+V_4}V_1V_2 & -\frac{1}{1+V_4}V_1V_3 & V_1 \\ -\frac{1}{1+V_4}V_2V_1 & 1-\frac{1}{1+V_4}V_2^2 & -\frac{1}{1+V_4}V_2V_3 & V_2 \\ -\frac{1}{1+V_4}V_3V_1 & -\frac{1}{1+V_4}V_3V_2 & 1-\frac{1}{1+V_4}V_3^2 & V_3 \\ -V_1 & -V_2 & -V_3 & V_4 \end{bmatrix}$$
$$\mapsto M \cdot (\cos A + i \sin A), \tag{3.1}$$

where $\cos A$ and $\sin A$ are given by (2.4) and (2.3). This is well defined because the decomposition of any matrix from $O_+^{\uparrow}(1,3)$ as product of space rotation and a boost is unique. Moreover, it is a bijection. In the following theorem is constructed effectively such an isomorphism [10].

Theorem 1. *The mapping (3.1) defines (local) isomorphism between the groups $O_+^{\uparrow}(1,3)$ and $SO(3,\mathbb{C})$.*

Indeed, the mapping

$$\begin{bmatrix} 0 & c & -b & ix \\ -c & 0 & a & iy \\ b & -a & 0 & iz \\ -ix & -iy & -iz & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & c+iz & -b-iy \\ -c-iz & 0 & a+ix \\ b+iy & -a-ix & 0 \end{bmatrix}$$

defines an isomorphism between the Lie algebras $o(1,3)$ and $o(3,\mathbb{C})$. This isomorphism induces local isomorphism between $O_+^{\uparrow}(1,3)$ and $SO(3,\mathbb{C})$, and it induces (local) isomorphism between the two groups. Further it is proved that this (local) isomorphism is given by (3.1).

If we want to find the composition of two space-time transformations which determine space rotations and velocities, there are two possibilities which lead to the same result: to multiply the corresponding two matrices from $SO(3,\mathbb{C})$ or from $O_+^\uparrow(1,3)$. Since the result is the same, the three dimensionality of the time is difficult to detect, and we feel like the time is 1-dimensional. The essential difference in using these two methods is the following. The Lorentz transformations give relationship between the coordinates of a 4-vector with respect tSo they show how the coordinates of a considered 4-vector change by changing the base space. On the other side, the matrices of the isomorphic group $SO(3,\mathbb{C})$ show how the space rotation and velocity change between two bodies, using the chosen base space, by consideration of changes in the fiber. So we have a duality in the SR. The use of the group $SO(3,\mathbb{C})$ alone is not sufficient, because their matrices are only Jacobi matrices free from any motion.

4 Preparation for the Theorem 3

(i) In the next section we want to deduce the Lorentz transformations using the group $SO(3,\mathbb{C})$. We assume that there is no effective motion, i.e. change of the basic coordinates, but simply rotation for an imaginary angle. Such a transformation will be called *passive motion*. The examination of observation of a moving body can easily be done in the following way.

Let us assume that $v_x = v$, $v_y = v_z = 0$. In this case the matrix $V = \cos A$ determined by (2.4) is given by

$$V = \cos A = \mathrm{diag}(1, \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}).$$

Hence there is no length contraction in the direction of motion (x-direction), while the lengths in any direction orthogonal to the direction of motion (yz-plane) are observed to be larger $(1-\frac{v^2}{c^2})^{-1/2}$ times. Notice that if we multiply all these length coefficients by $\sqrt{1-\frac{v^2}{c^2}}$ we obtain the prediction from the SR. Hence the observations for lengths for passive and active motions together is in agreement with the classical known results.

If there is an *active motion*, i.e. there is change of the basic coordinates, independently from the previous effect, we have the following phenomena. The group $SO(3,\mathbb{C})$ preserves the quantity $(\Delta x+ic\Delta t_x)^2+(\Delta y+ic\Delta t_y)^2+(\Delta z+ic\Delta t_z)^2$ and hence also preserves $\Delta x^2+\Delta y^2+\Delta z^2-c^2(\Delta t_x^2+\Delta t_y^2+\Delta t_z^2)=\Delta x^2+\Delta y^2+\Delta z^2-c^2\Delta t^2$.

(ii) The previous conclusion for the spatial lengths can be supported by the following conclusion about time intervals. While the observation of lengths may be done in different directions, the observation of time flow does not depend on the direction, but only on velocity. The time observed in a moving system is slower for coefficient $\sqrt{1-\frac{v^2}{c^2}}$ for active motions (analogously to the lengths). It is a consequence of the relativistic law of adding collinear velocities and it is presented by the following theorem ([10]).

Theorem 2. *Assume that the relativistic law of summation of collinear velocities is satisfied, and assume that the observed time in a moving inertial coordinate system with velocity v is observed to be multiplied with $f(\frac{v}{c})$, where f is a differentiable function and the first order Taylor development of f does not contain linear summand of v/c. Then, f must be $f(\frac{v}{c})=\sqrt{1-\frac{v^2}{c^2}}$.*

Since the 1-dimensional time direction is parallel to the velocity vector, there is no change in the observation of the time vector which corresponds to the passive motion. So the observed change for the time vector considered in the previous theorem comes only from the active motion.

Using the Theorem 2 and the assumption that the 1 - dimensional time is a quotient between the 3-vector of displacement and the 3-vector of velocity, the following conclusion is deduced in [10]. Let the initial and the end point of a 4-vector $\vec{r'}$ be simultaneous in one coordinate system S'. Then these two points in another coordinate system differ for time

$$\delta t = \frac{\frac{\vec{r'}\vec{v}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{4.1}$$

where $\vec{v}$ is the velocity vector. Notice that (4.1) is also a consequence from the Lorentz transformations.

(iii) The base manifold $\mathbb{R}^3$ is 3-dimensional. It is convenient to consider it as a subset of $\mathbb{C}^3$, consisting of

$$(x, y, z, ct_x, ct_y, ct_z),$$

where $ct_x = ct_y = ct_z = 0$ at a chosen initial moment, and call it complex base. The change of the coordinates can be done via the 6×6 real matrix $\begin{bmatrix} M\cos A & -M\sin A \\ M\sin A & M\cos A \end{bmatrix}$, where M is a space rotation. It acts on the 6-dimensional vectors $(\Delta x, \Delta y, \Delta z, 0, 0, 0)^T$ of the introduced complex base. Multiplying the vectors of the complex base $(\Delta x, \Delta y, \Delta z, 0, 0, 0)^T$ from left with this matrix, we obtain 3-dimensional base subspaces as they are viewed from the observer who rests with respect to the chosen complex base. Moreover, the pair $((\Delta x, \Delta y, \Delta z, 0, 0, 0)^T, G) \in \mathbb{R}^6 \times SO(3,\mathbb{C})$ viewed for moving and rotated base space determined by the matrix $P \in SO(3,\mathbb{C})$ is given by

$$(P(\Delta x, \Delta y, \Delta z, 0, 0, 0)^T, PGP^T) \in \mathbb{R}^6 \times SO(3,\mathbb{C}).$$

(iv) Until now we considered mainly the passive motions, while our goal is to consider active motion in the basic coordinates. The active motion is simply translation in the basic space, caused by the flow of the time. So besides the complex rotations of $SO(3,\mathbb{C})$ we should consider also translations in $\mathbb{C}^3$. Now $(\Delta ct_x, \Delta ct_y, \Delta ct_z)$ for the basic coordinates is not more a zero vector. The time which can be measured in basic coordinates is $\Delta t = [(\Delta t_x)^2 + (\Delta t_y)^2 + (\Delta t_z)^2]^{1/2}$. In case of motion of a point with velocity $\vec{v}$ we have translation in the basic coordinates for the vector $\vec{v}\Delta t + i\vec{c}\Delta t$. The space part $\vec{v}\Delta t$ is obvious, while the time part $\vec{c}\Delta t$ follows from (2.6). An orthogonal complex transformation may be applied, if previously the basic coordinates are translated.

5 Lorentz transformations as transformations on $\mathbb{C}^3$

For the sake of simplicity we will omit the symbol "Δ" for space coordinates. Let the coordinates x_s, y_s, z_s are denoted by x', y', z' and let us denote $\vec{r} = (x, y, z)$ and $\vec{r'} = (x', y', z')$. It is of interest to see the form of the Lorentz boosts as transformations in $\mathbb{C}^3$, while the space rotations are identical in both cases.

Theorem 3. *The following transformation in* $\mathbb{C}^3$

$$(1-\frac{v^2}{c^2})^{-1/2}\begin{bmatrix} \vec{r'} \\ \vec{c}t' + \frac{\vec{v}\times\vec{r'}}{c} \end{bmatrix} = \begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}\begin{bmatrix} \vec{r} + \vec{v}(t+\delta t) \\ \vec{c}(t+\delta t) \end{bmatrix} \tag{5.1}$$

via the group $SO(3,\mathbb{C})$ *is equivalent to the transformation given by a Lorentz boost determined by the isomorphism (3.1).*

The coefficient $\beta = (1-\frac{v^2}{c^2})^{-1/2}$ is caused by the active motion (i). There is a translation in the basic coordinates for vector $(\vec{v}(t+\delta t), \vec{c}(t+\delta t))$, where δt is defined by (4.1). On the other side, according to (2.6) in the moving system we have the time vector $\vec{v} \times \vec{r'}/c$, which disappears in basic coordinates ($\vec{v}=0$).

Proof. Notice that if we consider a space rotation P, which applies to all triples, the system (5.1) remains covariant. Indeed, $\vec{r}, \vec{r'}, \vec{v}, \vec{c}, \vec{v} \times \vec{r'}$ transform as vectors, t and δt, which is defined by (4.1), transform as scalars, while $\cos A$ and $\sin A$ transform as tensors of rank 2. Hence, if we multiply from left with $\begin{bmatrix} P & O \\ O & P \end{bmatrix}$ the both sides of (5.1), we obtain

$$\beta \begin{bmatrix} P\vec{r'} \\ P\vec{c}t' + \frac{(P\vec{v})\times(P\vec{r'})}{c} \end{bmatrix} =$$

$$= \begin{bmatrix} P\cos AP^T & -P\sin AP^T \\ P\sin AP^T & P\cos AP^T \end{bmatrix} \begin{bmatrix} P\vec{r} + P\vec{v}(t+\delta t) \\ P\vec{c}(t+\delta t) \end{bmatrix}$$

and since
$P(\cos A)P^T = \cos(PAP^T)$ and $P(\sin A)P^T = \sin(PAP^T)$, the covariance of (5.1) is proved. So it is sufficient to apply such a transformation P which maps vector $\vec{v}$ into $(v,0,0)$ and to prove the theorem in this special case.

Notice that both left and right side of (5.1) are linear functions of $x, y, z, t,\ x', y', z', t'$, and so after some transformations it can be simplified. Then the first three equations of (5.1) reduce to the following three equations respectively

$$x' = \frac{x+vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z.$$

Further, using these three equations, the fourth equation of (5.1) reduces to

$$t' = \frac{t + \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}},$$

while the 5-th and the 6-th equations are identically satisfied. □

According to Theorem 3 the well known 4-dimensional space-time is not fixed in 6 dimensions, but changes with the direction of velocity. Namely this 4-dimensional space-time is generated by the basic space vectors and the velocity vector from the imaginary part of the complex base.

Having in mind the equation (2.6), the Lorentz transformation (5.1) can be written in the following form

$$(1-\frac{v^2}{c^2})^{-1/2}\begin{bmatrix}\vec{r}_s\\ \vec{r}_t\end{bmatrix}=\begin{bmatrix}\cos A & -\sin A\\ \sin A & \cos A\end{bmatrix}\begin{bmatrix}\vec{r}+\vec{v}(t+\delta t)\\ \vec{c}(t+\delta t)\end{bmatrix}. \tag{5.2}$$

The coefficient $\beta=(1-\frac{v^2}{c^2})^{-1/2}$ is caused by the active motion (i). Since the coordinates $x_s, y_s, z_s, x_t, y_t, z_t$ are measured according to the basic coordinates x, y, z, we know that all of them are observed contracted for coefficient $\sqrt{1-\frac{v^2}{c^2}}$ so we needed to multiply them with $\beta=(1-\frac{v^2}{c^2})^{-1/2}$. If we denote again the same coordinates but now measured from the self coordinate system, then the Lorentz transformation becomes

$$\begin{bmatrix}\vec{r}_s\\ \vec{r}_t\end{bmatrix}=\begin{bmatrix}\cos A & -\sin A\\ \sin A & \cos A\end{bmatrix}\begin{bmatrix}\vec{r}+\vec{v}(t+\delta t)\\ \vec{c}(t+\delta t)\end{bmatrix}. \tag{5.3}$$

Analogous result can be obtained in case if there is simultaneously motion with a velocity and space rotation. If there is also rotation determined by the matrix $P\in SO(3,\mathbb{R})$, then the matrix $\begin{bmatrix}\cos A & -\sin A\\ \sin A & \cos A\end{bmatrix}$ should be replaced by $\begin{bmatrix}P\cos A & -P\sin A\\ P\sin A & P\cos A\end{bmatrix}$.

6 Cosmology based on the 3+3+3-model

According to (5.3) both vectors $\begin{bmatrix} \vec{r}_s \\ \vec{r}_t \end{bmatrix}$ and $\begin{bmatrix} \vec{r} + \vec{v}(t+\delta t) \\ \vec{c}(t+\delta t) \end{bmatrix}$ lie in the same 6-dimensional space. This is true, if the space-time of the Universe globally is Euclidean space with dimension 6. But, however, this is not in accordance with the temporary cosmology. Neglecting the time coordinates, it is accepted to be a 3-sphere. We shall modify this statement, by identifying the antipode points of the 3-sphere, and hence obtaining the projective space RP^3. This space is homeomorphic with the Lie group $SO(3,\mathbb{R})$. The local coordinates of $SO(3,\mathbb{R})$ are angles, i.e. real numbers, but we use length units for our local space coordinates. So for each small angle φ of rotation in a given direction corresponds coordinate length $R\varphi$ in the same direction, where R is a constant which can be called radius of the Universe. By accepting this modification of the spatial part of the Universe, we do not change anything locally, because the 3-sphere has locally the group structure of the unit quaternions, and locally this group is isomorphic with the group $SO(3,\mathbb{R})$. According to this small modification of the Universe it is now natural to assume that the space time of the Universe is isomorphic to $SO(3,\mathbb{C})$.

Notice that analogously as $(x+ict_x, y+ict_y, z+ict_z)$ is a local coordinate neighborhood of $SO(3,\mathbb{C})$, i.e. the space-time of the Universe, also $(1-\frac{v^2}{c^2})^{-1/2}(x_s+ix_t, y_s+iy_t, x_z+iz_t)$ is a coordinate neighborhood of the same manifold. These two coordinate systems can be considered as coordinate neighborhoods of $SO(3,\mathbb{C})$ as a complex manifold, because according to (5.3) the Cauchy-Riemannian conditions for these two systems are satisfied.

Let us discuss the space-time dimensionality of the Universe. We mentioned at the beginning of this section that the dimensionality is 6 if the space-time is flat. But now we have that it is parameterized by the following 9 independent coordinates: x, y, z coordinates which locally parameterize the spatial part of the Universe $SO(3, \mathbb{R})$, and $x_s, y_s, z_s, x_t, y_t, z_t$ coordinates which parameterize the bundle. Dually, the partial derivatives of these coordinates with respect to x, y, z lead to the same manifold, but now as a group of transformations. So in any case the total space-time of the Universe is homeomorphic to $SO(3, \mathbb{R}) \times SO(3, \mathbb{C})$, i.e. $SO(3, \mathbb{R}) \times \mathbb{R}^3 \times SO(3, \mathbb{R})$. In the above parameterization, $\mathbb{R}^3$ is indeed the space of velocities such that $|\vec{v}| < c$. Moreover, the group $SO(3, \mathbb{C})$ as well as the isomorphic group $O_+^{\uparrow}(1, 3)$ considers only velocities with magnitude less than c. If $|\vec{v}| = c$, then we have a singularity.

Notice that if we know the coordinates x, y, z and also $x_s, y_s, z_s, x_t, y_t, z_t$, then according to the Lorentz transformations, the time coordinates cx_t, cy_t, cz_t are uniquely determine. And conversely, if we know the coordinates x_s, y_s, z_s, and also $x, y, z, ct_x, ct_y, ct_z$, then the time coordinates x_t, y_t, z_t are uniquely determined. So we can say that **there are 6 spatial and 3 temporal coordinates**. Notice that if we consider that the Universe is a set of points, then it is more natural to consider it as 6-dimensional. But, since we consider the Universe as a set of orthonormal frames, so it is more natural to consider it as 9-dimensional.

This 9-dimensional space-time has the following property: **From each point of the space-time, each velocity and each spatial direction of the observer, the Universe seems to be the same**. In other words, there is no privileged space points (assuming that R is a global constant), no privileged direction and no privileged velocity. In other words, everything is relative.

Finally we can conclude the following. In the (total) space-time $SO(3,\mathbb{R}) \times SO(3,\mathbb{C})$, i.e. $SO(3,\mathbb{R}) \times \mathbb{R}^3 \times SO(3,\mathbb{R})$ there are three essential fiber bundles: (i) First case where the base is the space part of the Universe $SO(3,\mathbb{R})$ and the fiber is the group of orthogonal transformations $SO(3,\mathbb{C})$, which consists of all rotations and motions with velocities; (ii) second case where the base is the set of all spatial rotations $SO(3,\mathbb{R})$, and the fiber is the space-time part of the Universe $SO(3,\mathbb{C})$; (iii) third case where the base is the time part of the Universe $\mathbb{R}^3$ and the fiber is the set of space part of the universe and all space rotations, i.e. $SO(3,\mathbb{R}) \times SO(3,\mathbb{R})$.

The first and the second fiber bundles are direct products of the base $SO(3,\mathbb{R})$ and the fiber $SO(3,\mathbb{C})$. The first bundle was studied in many details in the first sections, while the second bundle was studied mainly in this section, after the topological structure of the space-time of the Universe was accepted. Here we want to emphasize that if we want to consider the fiber over a point B instead of the fiber over a point A (case (i)), then there is unique matrix $P \in SO(3,\mathbb{R})$, which maps the point A into point B, because both points A and B are matrices. In this case each matrix M from the fiber $SO(3,\mathbb{C})$ should be replaced with PMP^T. This implies that each 3-vector of velocity $\vec{v}$ should be replaced by $P\vec{v}$ and each space rotation M should be replaced by PMP^T. If A and B are relatively close points in the Universe, then P is practically unit matrix, and each matrix in the fiber remains unchanged. This is usually interpreted as translations in flat space-time as in Minkowski space. This case is close to the methods of the classical and relativistic mechanics. Indeed, it is sufficient to study the law of the change of the matrix M from the fiber, i.e. the matrix which consists the informations about the spatial rotation and the velocity vector of a considered test body. Then it is easy to find the trajectory of motion of the test body.

Now let us consider the case (ii). Let us choose an arbitrary space rotation. This may be done by arbitrary 3 orthonormal tangent vectors over the Universe considered as $SO(3,\mathbb{R})$ manifold, and then it can be transfered at each point of $SO(3,\mathbb{R})$, by using the group structure of $SO(3,\mathbb{R})$. Then the studying of motion of arbitrary test body means to find how changes the matrix which gives the position of the test body in the space and its velocity at the chosen moment. The space rotation of the test body also changes, analogously as the position of the test body in case (i) also changes, but in case (ii) we are not interesting about it. This is close to the methods of the quantum mechanics, where the coordinate operators (i.e. space coordinate operators) and the impulse operators (i.e. the time coordinate operators) have the main role. If we want to use another space rotation instead of the chosen one, then there exists a unique matrix P which maps the initial space rotation into the new space rotation. In this case each matrix M from the fiber $SO(3,\mathbb{C})$ should be replaced with PMP^T. This implies that each 3-vector of velocity $\vec{v}$ should be replaced by $P\vec{v}$ and each space position, which is given by orthogonal matrix S, should be replaced by PSP^T. Since the radius of the Universe is extremely large, practically for each space position, the matrix S for space position is close to the unit matrix, and approximately can be considered as a vector of translation $\vec{r}$, and now this vector should be replaced by $P\vec{r}$.

While in the first and the second case the group and the fiber is simultaneously $SO(3,\mathbb{C})$, in the third case the fiber and the group of the fiber is $SO(3,\mathbb{R}) \times SO(3,\mathbb{R})$, which is given by $(X,Y)(X',Y') = (XX',YY')$. So, while the time coordinates are functionally dependnt with the both sets of spatial coordinates, both sets of spatial coordinates mutually independent.

The previous discussion can be summarized by the following diagram,

$$\begin{array}{ccc} & \mathrm{V} \cong \mathbb{R}^3 & \\ \times & & \times \\ \mathrm{S} \cong SO(3,\mathbb{R}) & \times & \mathrm{SR} \cong SO(3,\mathbb{R}) \end{array}$$

consisting of three 3-dimensional sets: velocity (V) which is homeomorphic to $\mathbb{R}^3$, space (S) which is homeomorphic to $SO(3,\mathbb{R})$, and space rotation (SR) which is homeomorphic to $SO(3,\mathbb{R})$. Each of these three sets can be considered as base, while the Cartesian product of the other two sets can be considered simultaneously as a fiber and also a group which acts over the fiber. The first set is not a group and must be joined with each of the other two sets, while the other two sets may exist independently because they are groups.

7 Modification of the Lorentz transformations

By acceptance a priori of the previous assumptions, as a consequence we do not have any translations and vectors. So the classical Lorentz transformations have no sense any more. From this viewpoint, we should modify the Lorentz transformation defined on a flat space. The vectors $\begin{bmatrix} \vec{r}_s \\ \vec{r}_t \end{bmatrix}$

and $\begin{bmatrix} \vec{r} + \vec{v}(t+\delta t) \\ \vec{c}(t+\delta t) \end{bmatrix}$ should be replaced by two matrices and the Lorentz transformation will become a matrix equality in the following way. Notice that each small neighborhood of the unit matrix in $SO(3,\mathbb{C})$ can be parameterized by the following 6-dimensional vector (x, y, z, x', y', z'), i.e. vector $(z_1, z_2, z_3) \in \mathbb{C}^3$, where $z_1 = x + ix'$, $z_2 = y + iy'$, $z_3 = z + iz'$, by joining the following matrix in $SO(3,\mathbb{C})$

$$\left\{ \frac{1}{\Delta} \begin{bmatrix} 1 + z_1^2 - z_2^2 - z_3^2 & -2z_3 + 2z_1z_2 & 2z_2 + 2z_1z_3 \\ 2z_3 + 2z_1z_2 & 1 - z_1^2 + z_2^2 - z_3^2 & -2z_1 + 2z_2z_3 \\ -2z_2 + 2z_1z_3 & 2z_1 + 2z_2z_3 & 1 - z_1^2 - z_2^2 + z_3^2 \end{bmatrix} \right\}^{1/2},$$

where $\Delta = 1 + z_1^2 + z_2^2 + z_3^2$. In a special case, if $x' = y' = z' = 0$, this matrix represents a spatial rotation in the direction of $(x, y, z) = (z_1, z_2, z_3)$ and the angle of rotation is $\varphi = \arctan\sqrt{x^2 + y^2 + z^2}$. This representation can be extended to the following special case, which will be used later. If $\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = \frac{v}{c}$, then the direction of rotation is $(x, y, z) = \frac{v}{c}(x', y', z')$ and the complex angle of rotation satisfies

$$\tan\varphi = \sqrt{(x + ix')^2 + (y + iy')^2 + (z + iz')^2} =$$
$$= \sqrt{x^2 + y^2 + z^2}(1 + i\frac{c}{v}).$$

For example, if $\vec{v} = (0, 0, v)$, then the corresponding orthogonal matrix is given by

$$\begin{bmatrix} \cos\frac{z}{R} & -\sin\frac{z}{R} & 0 \\ \sin\frac{z}{R} & \cos\frac{z}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\frac{z'}{R} & -i\sinh\frac{z'}{R} & 0 \\ i\sinh\frac{z'}{R} & \cosh\frac{z'}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} \cosh\frac{z'}{R} & -i\sinh\frac{z'}{R} & 0 \\ i\sinh\frac{z'}{R} & \cosh\frac{z'}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{z}{R} & -\sin\frac{z}{R} & 0 \\ \sin\frac{z}{R} & \cos\frac{z}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (7.1)$$

The equality (5.2) now can be represented in the following way. First note that (5.2) can be written in the following form

$$\frac{1}{R}\begin{bmatrix}\vec{r}_s\\ \vec{r}_t\end{bmatrix}=\frac{1}{R}\begin{bmatrix}\cos A & -\sin A\\ \sin A & \cos A\end{bmatrix}\begin{bmatrix}\vec{r}\\ 0\end{bmatrix}\sqrt{1-\frac{v^2}{c^2}}+$$

$$+\frac{1}{R}\begin{bmatrix}\vec{v}\\ \vec{c}\end{bmatrix}(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}.$$

Both vectors $\frac{1}{R}\begin{bmatrix}\vec{r}_s\\ \vec{r}_t\end{bmatrix}$ and $\frac{1}{R}\begin{bmatrix}\cos A & -\sin A\\ \sin A & \cos A\end{bmatrix}\begin{bmatrix}\vec{r}\\ 0\end{bmatrix}\sqrt{1-\frac{v^2}{c^2}}$ should be replaced by two matrices M' and M from $SO(3,\mathbb{C})$, while the vector $\frac{1}{R}\begin{bmatrix}\vec{v}\\ \vec{c}\end{bmatrix}(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}$ should be replaced by the following 3×3 complex orthogonal matrix

$$L\begin{bmatrix}\cos\frac{v(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & -\sin\frac{v(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & 0\\ \sin\frac{v(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & \cos\frac{v(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & 0\\ 0 & 0 & 1\end{bmatrix}\times$$

$$\times\begin{bmatrix}\cosh\frac{c(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & -i\sinh\frac{c(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & 0\\ i\sinh\frac{c(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & \cosh\frac{c(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & 0\\ 0 & 0 & 1\end{bmatrix}L^T,$$

which obtains from (7.1) by replacing $z=v(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}$ and $z'=c(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}$, and where $L\in SO(3,\mathbb{R})$ is arbitrary orthogonal matrix which maps the vector $(0,0,1)^T$ into the vector $(v_x/v,v_y/v,v_z/v)^T$. Hence the Lorentz transformation (5.2) takes the following matrix from

$$M' = ML \begin{bmatrix} \cos \frac{v(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & -\sin \frac{v(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & 0 \\ \sin \frac{v(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & \cos \frac{v(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix} \times$$

$$\times \begin{bmatrix} \cosh \frac{c(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & -i \sinh \frac{c(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & 0 \\ i \sinh \frac{c(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & \cosh \frac{c(t+\delta t)\sqrt{1-\frac{v^2}{c^2}}}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix} L^T, \quad (7.2)$$

and now there are no translations. Notice that (7.2) is not equivalent to (5.2). But neglecting the terms of order R^{-2} and less, which are extremely small, the equalities (5.2) and (7.2) are equivalent. However, we accept now that (7.2) is exact equality, while (5.2) is approximative.

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