Supersymmetry and All That :
A simple example in 1–dimension

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Outline :

- Very brief history
- N=1 SUSY in 1 dimension
    - Component formalism
    - Superfield formalism
- N=2 SUSY in 1 dimension
    - How to extend
    - An alternative way of obtaining the action

Supersymmetry

- 1967 Coleman-Mandula no-go theorem:
  It is not possible to extend the Poincare group $(P_\mu, J_{\mu\nu})$ in a non-trivial way (i.e. the only way :$[P_\mu, \Omega] = 0 = [J_{\mu\nu}, \Omega]$)

- 1971 Golfand-Likhtman - Birth of SUSY
  If a Lie group has a graded structure it is possible to extend the Poincare group. $\Rightarrow$ Superalgebra (Z_2 graded structure)

$$SUSY \,:\, Fermion \rightarrow Boson \,,\; Boson \rightarrow Fermion$$

- 1970's Superstring theories

- 1974 Wess-Zumino Model
  First renormalizable theory in 4-dim.s $\Rightarrow$ SUSY becomes popular

- 1975 Haag-Lopusanski-Sohnius
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Supersymmetric theories are highly restricted :

- Bosons and fermions can only be related to each other by fermionic symmetry operators Q of spin–1/2 (not spin–3/2 or higher).

$$Q|fermion >= |boson > \quad , \quad Q|boson >= |fermion >$$

- Only in the presence of SUSY, multiplets can contain particles of different spin.
- Particles in the same supermultiplet have the same mass and coupling constant.
- no. of bosons = no. of fermions in a supersymmetric theory.

One can write SUSY transformations and supersymmetric actions in two different ways :

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- Write the most general action that we know from field theory including kinetic, mass, interaction terms.
- Fix the coefficients in the action so that it is invariant under SUSY.
- This is a tedious but a straightforward way to construct. *(See for instance book by West for details.)*

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- Supersymmetric actions can then be written directly in terms of SUPERFIELDS $\Phi(x, \theta, \bar{\theta})$ and their super derivatives.
- This is an elegant way to construct but may not work for every case. *(See any book on SUSY for details.)*

A Simple Example in 1–Dimension :

Let us consider,

- A real scalar field $\phi(t)$,
- A real fermionic field $\psi(t)$.

For each t, $\psi(t)$, is an independent Grassmann variable :

$$\psi(t_1)\psi(t_2) = -\psi(t_2)\psi(t_1) \Rightarrow (\psi(t))^2 = 0$$

Assume that $d\psi(t)/dt \equiv \dot{\psi}(t)$ exists then we also have

$$\dot{\psi}(t_1)\dot{\psi}(t_2) = -\dot{\psi}(t_2)\dot{\psi}(t_1)\,,\ \dot{\psi}(t)\psi(t) = -\psi(t)\dot{\psi}(t)$$

Therefore ψ and $\dot{\psi}$ anticommute with themselves and with each other at equal time t :

$$\{\psi(t),\psi(t)\} = 0\,,\ \{\psi(t),\dot{\psi}(t)\} = 0\,,\ \{\dot{\psi}(t),\dot{\psi}(t)\} = 0$$

Note the difference between ϕ and ψ

$$\int dt\phi(t)\dot{\phi}(t) = \frac{1}{2}\int dt\frac{d}{dt}(\phi(t)\phi(t))$$

but

$$\int dt\psi(t)\dot{\psi}(t) \neq \frac{1}{2}\int dt\frac{d}{dt}(\psi(t)\psi(t))$$

As an action for these fields we take,

$$I = \int dt \left(\frac{1}{2}\dot{\phi}^2 + \frac{i}{2}\psi\dot{\psi} \right).$$

Note that in this action,

- $\frac{1}{2}\dot{\phi}^2$ term is a truncation of the Klein-Gordon action to an (x, y, z) independent field,
- $\frac{i}{2}\psi\dot{\psi}$ term is a truncation of the Dirac action for a real spinor to one of its component that is also independent of (x, y, z).

For $\hbar = 1$ the dimension of the action is zero ($[I] = 0$), therefore taking $[t] = -1$ we get the dimensions of the fields as

$$[\phi] = -\frac{1}{2}\,,\ [\psi] = 0$$

SUSY Transformations

We look for a symmetry such that

$$\begin{aligned} \delta_\xi(boson) &= \xi(something fermionic) \\ \delta_\xi(fermion) &= \xi(something bosonic) \end{aligned}$$

It is clear that ξ must be anticommuting !

- Let $\delta_\xi \phi = i\xi\psi$. Then $[\xi] = -1/2$.

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- If the transformation is linear the only possible choice is $f \sim \xi\dot{\phi}$.

  Indeed, in order to get $\delta_\xi I = 0$ we find

$$\delta_\xi \psi = -\xi\dot{\phi}.$$

Summary :

- SUSY transformations : $\delta_\xi \phi = i\xi\psi \,,\; \delta_\xi \psi = -\xi\dot{\phi}$.
- $I = \int dt \left(\frac{1}{2}\dot{\phi}^2 + \frac{i}{2}\psi\dot{\psi} \right)$ is invariant (i.e. superysmmetric).
- $(\phi \,,\, \psi)$ real scalar supermultiplet.
- # of bosons = # of fermions

SUSY Algebra :

To get the algebra let us study the commutator of two SUSY transformations:

$$[\delta_\eta, \delta_\xi]\phi = (\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi = 2i\eta\xi\dot{\phi}$$

$$[\delta_\eta, \delta_\xi]\psi = 2i\eta\xi\dot{\psi}$$

We get,

$$[\delta_\eta, \delta_\xi] = 2i\eta\xi\left(\frac{d}{dt}\right)$$

for constant parameters ξ, η.
Note that right hand side is a translation over a distance $t_0 = 2i\eta\xi$!

Let us obtain the algebra in a tricky way. We have

$$i\frac{d}{dt} \equiv H\,,\ \delta_\xi \equiv i\xi Q\,,\ \delta_\eta \equiv i\eta Q$$

and H ve Q denotes the generators of translation and SUSY.
The commutator of SUSY transformations can be written in terms of anticommutators as

$$[\delta_\eta, \delta_\xi] = -(\xi Q\eta Q - \eta Q\xi Q) = \xi\eta(QQ + QQ) = \xi\eta\{Q, Q\}$$

With the help of above definitions we get

$$\{Q, Q\} = 2H$$

Jacobi identity gives the other relation :

$$[Q, H] = 0$$

Therefore, SUSY algebra in one dimension is

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- Note that, algebra contains both commutators and anticommutators,
- therefore, Q and H generators form a graded Lie algebra as promised before.

REMARK :

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  $\Rightarrow$ prediction : for every fermionic particle there should be a fermionic antiparticle .
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- 1970's $\sqrt{H} \sim Q$: Supersymmetry
  $\Rightarrow$ prediction : for every particle there should be a superpartner .
  (201?, will LHC find one?)

Grassmann Algebra :

Let $\theta_i\,, i = 1, 2 \cdots n$ to be n Grassmann numbers that satisfies,

$$\theta_i\theta_j + \theta_j\theta_i = 0\,,\, \forall\, i, j \to \theta_i\theta_i = 0$$

Definition of derivative and integration is given as

$$\frac{\partial}{\partial\theta_i}\theta_j = \delta_{ij}\ ,\ \int d\theta_1 d\theta_2 \cdots d\theta_n\theta_n \cdots \theta_2\theta_1 = 1$$

Note that integral operates as a derivative!
The above relations simplify a lot for one θ :

$$\int d\theta\theta = 1\,,\, \int d\theta c = 0 \to \int d\theta \equiv \frac{d}{d\theta}$$

Superspace :

Since in one dimension we have one Q and H we parametrize the space with θ and t where it is obvious that both are real and θ is Grassman variable.

DEFINITION :

- The space with coordinates t and θ is called SUPERSPACE.
- Any function of t and θ (i.e. $\Phi(t,\theta)$) is called a SUPERFIELD.

Then since, $\theta^2 = 0$ in our model the simplest superfield is,

$$\Phi(t, \theta) = \phi(t) + i\theta\psi(t)$$

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- All components of Φ must be scalar. Therefore, since θ is a Grassmann variable, ψ must be a anticommuting field.
- All components of Φ must be real.
  Thats why we have an i in front $\theta\psi$.
- Since $[\Phi] = [\phi] = -1/2$ we must also have $[\psi] = 0$ and $[\theta] = -1/2$.

SUSY transformations can be obtained with the help of a Hermitian operator,

$$Q = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}$$

because

$$\xi Q\Phi = \xi\left(\frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}\right)\Phi = i\xi\psi + i\theta(-\xi\dot{\phi}) = \delta\phi + i\theta\delta\psi$$

Note that this operator Q satisfies SUSY algebra :

$$\{Q, Q\} = 2i\frac{\partial}{\partial t} \equiv H\,,\ [Q, H] = 0\,,\ [H, H] = 0$$

The invariance of the action can be written as

$$\delta I = \int dtd\theta\xi Q[\cdots] = 0.$$

It is important to know other operators that commute with ξQ. One of them is d/dt'dir. The other one is defined as

$$D = \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t}$$

and it is called super covariant derivative.
From the definition we see that,

$$[\xi Q, D] = \xi\{Q, D\} = 0.$$

Let us construct the action by using *"what else can it be"* method. In general we can write,

$$I = \int dt d\theta F(\frac{\partial}{\partial t}, D, \Phi).$$

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$$F = K(\frac{\partial}{\partial t}, D).\Phi^2.$$

- Since $[\Phi] = -1/2$ we can only have $[K] = 3/2$. So only by doing dimensional analysis we get only one solution for K:

$$K \sim \frac{\partial}{\partial t}.D.$$

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- which gives

$$\begin{aligned} I &= \frac{i}{2}\int dtd\theta \left(\dot{\phi} + i\theta\dot{\psi}\right)\left(i\psi - i\theta\dot{\phi}\right) = 0 + \frac{i}{2}\int dtd\theta \left(-i\theta\dot{\phi}\dot{\phi} + \theta\psi\dot{\psi}\right) + 0 \\ &= \int dtd\theta \left(\frac{1}{2}\theta\dot{\phi}\dot{\phi} + \frac{i}{2}\theta\psi\dot{\psi}\right) \end{aligned}$$

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- This is the same action that we obtained before without using superspace techniques !

Extended SUSY :

Now let us consider more then one SUSY in one dimension, i.e. we have N SUSY generators Q^i , $i = 1, 2, \cdots, N$ so that

$$\delta_\xi \phi = \xi_i Q_i \phi = i \sum_{i=1}^{N} \xi_i \psi_i \quad , \quad \delta_\xi \psi = \xi_i Q_i \psi = -\xi_i \dot{\phi}.$$

The action,

$$I = \int dt \left(\frac{1}{2} \dot{\phi} \dot{\phi} - \frac{i}{2} \sum_{i=1}^{N} \psi_i \dot{\psi}_i \right)$$

is still invariant under this extended SUSY transformations.

But there is something unusual

There are N fermions but still 1 boson !

Let us consider N=2 and check SUSY algebra,

$$[\delta_\xi, \delta_\eta]\phi = 2i(\xi_1\eta_1 + \xi_2\eta_2)\dot{\phi}$$

as expected but for instance for ψ_1 we get

$$[\delta_\xi, \delta_\eta]\psi_1 = 2i\xi_1\eta_1\dot{\psi}_1 + i(\xi_1\eta_2 + \xi_2\eta_1)\dot{\psi}_2$$

and it doesn't close on translation of ψ_1 unless we use equation of motion of ψ_2 : $\dot{\psi}_2 = 0$.

Such a SUSY called *onshell SUSY*

Since we have 2 fermions ψ_1, ψ_2 and one boson ϕ, and algebra doesn't close automatically to cure the problem let us introduce another boson field F such that

$$I = \int dt \left(\frac{1}{2}\dot{\phi}\dot{\phi} - \frac{i}{2}\psi_i\dot{\psi}_i + \frac{1}{2}F^2 \right)$$

Note that F doesn't have a kinetic term and it is called *auxiliary field*.

One can view (ϕ, ψ_1) as one multiplet and (ψ_2, F) as another such that

$$\delta\phi = i\xi_1\psi_1 \, , \, \delta\psi_1 = -\xi_1\dot{\phi} \quad , \quad \delta\psi_2 = \xi_2 F \, , \, \delta F = i\xi_2\dot{\psi}_2$$

with

$$I_1 = \int dt \left(\frac{1}{2}\dot{\phi}\dot{\phi} - \frac{i}{2}\psi_1\dot{\psi}_1 \right) \quad , \quad I_2 = \int dt \left(-\frac{i}{2}\psi_2\dot{\psi}_2 + \frac{1}{2}F^2 \right)$$

i.e. $N = 2 = 1 + 1$

However, one can construct N=2 SUSY as N=2 ! (Like for the 4-dimensional case.)
Let us write more general SUSY transformations by analyzing dimensions of the fields.

$$\delta\phi = i\xi_i\psi_i \quad , \quad \delta\psi_i = \xi_i\dot{\phi} + \alpha_{ij}\xi_j F \quad , \quad \delta F = i\xi_i\beta_{ij}\dot{\psi}_j$$

where α and β are real matrices.

From the commutator algebra

$$[\delta_\xi, \delta_\eta]\phi = 2i\eta_i\xi_i\dot{\phi} + (i\eta_i(\alpha_{ij} + \alpha_{ji})\xi_j F)$$

$$[\delta_\xi, \delta_\eta]\psi_i = i(\eta_i\xi_j - \xi_i\eta_j)\dot{\psi}_j + i\alpha_{ij}\beta_{kl}(\eta_j\xi_k - \xi_j\eta_k)\dot{\psi}_l$$

we get

- $\alpha_{ij} + \alpha_{ji} = 0 \quad , \quad \alpha_{ij}\beta_{jk} = \delta_{ik}$

so that $[\delta_\xi, \delta_\eta]$ closes to translations.

Finally, off–shell N=2 SUSY algebra in 1–dimension can be written as,

$$\delta\phi = i\xi_i\psi_i \quad , \quad \delta\psi_i = \xi_i\dot{\phi} + \epsilon_{ij}\xi_j F \quad , \quad \delta F = \xi_i\epsilon_{ij}\dot{\psi}_j$$

where $\epsilon^{12} = -\epsilon^{21} = 1$.

Moreover, since we have two fermions now we can write mass term

$$I_m = -m\int dt\,(F\phi + i\psi_1\psi_2)$$

and an interaction term

$$I_g = g\int dt\left(\frac{1}{2}F\phi^2 + i\psi_1\psi_2\phi\right)$$

An alternative way to obtain the action :

Note that

- Fields they belong to a supersymmetric multiplet.
- One can move from the lowest member of the multiplet to highest one by SUSY transformation.
- Since Action is supersymmetric it also belongs to a SUSY multiplet.
- Therefore, one should be able to obtain the action by applying multiple SUSY variations to a lower dimensional integrated field polynomial.

Of course, one can say that this observation is related with superspace. But,

- we need off–shell formulation to write superfields,
- and off–shell formulation does not always exist !

Let us show, how this method works for our simple model : We can write the SUSY transformation as,

$$\delta_\xi = \xi_1 Q_1 + \xi_2 Q_2$$

For simplicity let us also define

$$Q = \frac{Q_1 + iQ_2}{\sqrt{2}} \quad , \quad \bar{Q} = \frac{Q_1 - iQ_2}{\sqrt{2}} \quad , \quad \psi = \frac{\psi_1 + i\psi_2}{\sqrt{2}} \quad , \quad \bar{\psi} = \frac{\psi_1 - i\psi_2}{\sqrt{2}}$$

We can write Q and $\bar{Q}$ variations as

$$Q\phi = i\psi, \quad , \quad Q\psi = 0 \quad , \quad Q\bar{\psi} = \dot{\phi} - iF \quad , \quad QF = \dot{\psi}$$

$$\bar{Q}\phi = i\bar{\psi}, \quad , \quad \bar{Q}\bar{\psi} = 0 \quad , \quad \bar{Q}\psi = \dot{\phi} + iF \quad , \quad \bar{Q}F = -\dot{\bar{\psi}}$$

Then by analysis the dimensions of the fields and parameters,

$$[t] = -1, [m] = 1, [g] = \frac{3}{2}, [Q] = \frac{1}{2}, [\phi] = -\frac{1}{2}, [\psi] = 0, [F] = \frac{1}{2}$$

it is easy to get,

$$I = \int dt \left[-(\bar{Q}Q)^2\phi^2 + \bar{Q}Q\left(\frac{m}{2}\phi^2 + \frac{g}{3!}\phi^3\right)\right]$$

In other words, we can write the action by applying multiple super variations of the monomials of ϕ.

This also true for the on–shell transformations except that one gets the action modulo equation of motion of fermion fields!

A similar construction also works in 4 dimensions for
WZ, N=1 and N=2 SYM (K.U, MPLA'XX)
and even in much more complicated cases (H.Sonoda, K.U,2009).

Reference

- Nearly all of this talk is from the excellent lectures :
  P. van Nieuwenhuizen "Supersymmetry, Supergravity, Superspace and BRST Symmetry in a Simple Model " arXiv: hep-th/0408179
- The very minor part about cohomology is from some unpublished notes of mine, "An Introduction to SUSY", FGE 2005.
- One of the standard reference in 4–dimension is,
  J. Wess and J. Bagger, "Supersymmetry and Supergravity", (1992).