Membranes in Curved Superspace

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Challenges Beyond the Standard Model
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- Introduction
- On-shell 11d Superspace
- Theta-expansion
  Recursion relations
  All-order results
  Supermembrane theory: covariant vertex operators
- Conclusions

Based on: D.T. JHEP **0411** (2004) 087; hep-th/0407244

Introduction

- Perturbative String Theory $\longrightarrow$ M-theory
  Fundamental string $\longrightarrow$ supermembrane
- Direct Quantization:
  No conformal invariance, nonlinearities
- Finite-N regularization:
  Supersymmetric matrix quantum mechanics
  Continuous spectrum, membrane instability
- The BFSS matrix model
  M-theory in the IMF
  Curved backgrounds?
- Scattering amplitudes
  Light-cone, pure-spinors
- Membrane (M5) instantons
  Nonperturbative superpotentials
  Cosmology, KKLT...
- Component form of the wv action?

The eleven-dimensional supermembrane is given by the superembedding

$$Z : \Sigma^{(3|0)} \to M^{(11|32)}$$

with supercoordinates

$$Z^{\underline{M}} := (X^{\underline{m}}, \theta^{\underline{\mu}})$$

World-volume theory:

$$S = \int_{\Sigma} d\sigma^3 \{\sqrt{-g} + f^* C\}$$

where

$$f^* C := \frac{1}{6} \varepsilon^{mnp} \partial_m Z^{\underline{P}} \partial_n Z^{\underline{N}} \partial_p Z^{\underline{M}} C_{\underline{MNP}}$$

and

$$g_{mn} := \left(\partial_m Z^{\underline{M}} E_{\underline{M}}{}^{\underline{a}}\right)\left(\partial_n Z^{\underline{N}} E_{\underline{N}}{}^{\underline{b}}\right)\eta_{\underline{ab}}$$

Need the explicit θ-expansion of

$$E_{\underline{M}}{}^{\underline{A}}(X, \theta)$$

On-shell 11d supergravity in superspace

Flat supercoordinates

$$A = (a, \alpha), \qquad a = 0, 1, \ldots 9; \;\; \alpha = 1, \ldots 32$$

Torsion and curvature:

$$T^A = \nabla E^A := dE^A + E^B \Omega_B{}^A = \frac{1}{2} E^C E^B T_{BC}{}^A$$

$$R_A{}^B = d\Omega_A{}^B + \Omega_A{}^C \Omega_C{}^B = \frac{1}{2} E^D E^C R_{CD,A}{}^B$$

Bianchi identities:

$$\nabla T^A = E^B R_B{}^A$$

$$\nabla R_B{}^A = 0$$

CJS supergravity follows from:

$$T_{\alpha\beta}{}^a = -i(\gamma^a)_{\alpha\beta}$$

- The physical fields "sit" in the components of the torsion!

- The θ-expansion is generated by ∇_α

$$\Phi^{(n)} \sim (\nabla_\alpha)^n \Phi, \qquad @\, \theta = 0$$

Equations-of-motion

Spinorial derivatives:

$$\nabla_\alpha G_{abcd} = 6i(\gamma_{[ab}T_{cd]})_\alpha$$

$$\nabla_\alpha T_{ab}{}^\beta = \frac{1}{4}R_{ab,cd}(\gamma^{cd})_\alpha{}^\beta - 2\nabla_{[a}T_{b]\alpha}{}^\beta - 2T_{[a|\alpha}{}^\epsilon T_{|b]\epsilon}{}^\beta$$

$$\nabla_\alpha R_{ab,cd} = 2\nabla_{[a|}R_{\alpha|b]cd} - T_{ab}{}^\epsilon R_{\epsilon\alpha cd} + 2T_{[a|\alpha}{}^\epsilon R_{\epsilon|b]cd}$$

Equations-of-motion:

$$\nabla_{[a}G_{bcde]} = 0$$

$$\nabla^f G_{fabc} = -\frac{1}{2(4!)^2}\varepsilon_{abcd_1\ldots d_8}G^{d_1\ldots d_4}G^{d_5\ldots d_8}$$

$$(\gamma^a T_{ab})_\alpha = 0$$

$$R_{ab} - \frac{1}{2}\eta_{ab}R = -\frac{1}{12}(G_{adfg}G_b{}^{dfg} - \frac{1}{8}\eta_{ab}G_{dfge}G^{dfge})$$

Curved indices:

$$e_m{}^a e_n{}^b e_p{}^c e_q{}^d G^{(0)}_{abcd} = 4\partial_{[m}C^{(0)}_{npq]} - 6i(\Psi_{[m}\gamma_{np}\Psi_{q]})$$

$$\begin{aligned} e_m{}^a e_n{}^b\, T^{(0)}_{ab}{}^\alpha &= \partial_m \Psi_n{}^\alpha + \omega_{m\beta}{}^\alpha \Psi_n{}^\beta \\ &\quad + (\Psi_m \mathcal{T}_n{}^{abcd})^\alpha G^{(0)}_{abcd} - (m \leftrightarrow n) \end{aligned}$$

$$\begin{aligned} e_m{}^a e_n{}^b e_k{}^c e_l{}^d R^{(0)}_{abcd} &= R(\omega)_{mnkl} + i(\Psi_m \mathcal{R}_{kl}{}^{abcd}\Psi_n)G^{(0)}_{abcd} \\ &\quad - 2i(\Psi_{[m}\mathcal{S}_{n]kl}{}^{ab}T^{(0)}_{ab}) \end{aligned}$$

Gauge-fixing

Expand

$$S_{\{A\}} = \sum_{n=0}^{32} S^{(n)}_{\{A\}},$$

where

$$S^{(n)}_{\{A\}} := \frac{1}{n!}\theta^{\mu_n}\dots\theta^{\mu_1}S^{(n)}_{\mu_1\dots\mu_n,\{A\}}$$

Use superdiffeomorphisms and Lorentz transformations to set

$$E^{(n)}_{[\mu_1\dots\mu_n,\mu]}{}^{A} = 0$$
$$\Omega^{(n)}_{[\mu_1\dots\mu_n,\mu]A}{}^{B} = 0$$

Concisely:

$$\theta^{\mu}(E_{\mu}{}^{A} - \delta_{\mu}{}^{A}) = 0$$
$$\theta^{\mu}\Omega_{\mu A}{}^{B} = 0$$

Also:

$$\theta^{\mu}\partial_{\mu} = \theta^{\mu}\delta^{\alpha}_{\mu}\nabla_{\alpha}$$

and therefore

$$S^{(n)} = \frac{(n-r)!}{n!}\,\theta^{\alpha_r}\dots\theta^{\alpha_1}\,(\nabla_{\alpha_1}\dots\nabla_{\alpha_r}S)^{(n-r)}$$

Recursion

$$
\begin{aligned}
G^{(n)}_{abcd} &= \frac{6i}{n}(\theta\Gamma_{[ab}T^{(n-1)}_{cd]}) \\
T^{(n)}_{ab}{}^{\alpha} &= \frac{1}{4n}(\theta\Gamma^{cd})^{\alpha}R^{(n-1)}_{abcd} + \frac{2}{n}(\theta\mathcal{T}_{[a}{}^{cdef})^{\alpha}(\nabla_{b]}G_{cdef})^{(n-1)} \\
&\quad - \frac{2}{n}(\theta\mathcal{T}_{[a}{}^{cdef}\mathcal{T}_{b]}{}^{c'd'e'f'})^{\alpha}(G_{cdef}G_{c'd'e'f'})^{(n-1)} \\
R^{(n)}_{abcd} &= -\frac{2i}{n}(\theta\mathcal{S}_{[a|cd}{}^{ef})_{\alpha}(\nabla_{|b]}T_{ef}{}^{\alpha})^{(n-1)} \\
&\quad - \frac{i}{n}(\theta\mathcal{R}_{cd}{}^{efgh})_{\alpha}(T_{ab}{}^{\alpha}G_{efgh})^{(n-1)} \\
&\quad + \frac{2i}{n}(\theta\mathcal{T}_{[a}{}^{efgh}\mathcal{S}_{b]cd}{}^{e'f'})_{\alpha}(T_{e'f'}{}^{\alpha}G_{efgh})^{(n-1)}
\end{aligned}
$$

Note:

$$
\begin{aligned}
T^{(n)}_{ab}{}^{\alpha} &= \frac{i}{n(n-1)}\{(\mathcal{M}_{[a|}{}^{ef})^{\alpha}{}_{\beta}(\nabla_{|b]}T_{ef}{}^{\beta})^{(n-2)} \\
&\qquad + (\mathcal{N}_{ab}{}^{c_1\ldots c_6})^{\alpha}{}_{\beta}(G_{c_1\ldots c_4}T_{c_5c_6}{}^{\beta})^{(n-2)}\}
\end{aligned}
$$

where

$$
\begin{aligned}
(\mathcal{M}_{a}{}^{ef})^{\alpha}{}_{\beta} &:= -\frac{1}{2}(\theta\Gamma^{bc})^{\alpha}(\theta\mathcal{S}_{abc}{}^{ef})_{\beta} + 12(\theta\mathcal{T}_{a}{}^{bcef})^{\alpha}(\theta\Gamma_{bc})_{\beta} \\
(\mathcal{N}_{ab}{}^{c_1\ldots c_6})^{\alpha}{}_{\beta} &:= -\frac{1}{4}(\theta\Gamma^{ef})^{\alpha}(\theta\mathcal{R}_{ef}{}^{c_1\ldots c_4})_{\beta}\delta^{c_5}_{[a}\delta^{c_6}_{b]} + \ldots
\end{aligned}
$$

More recursion

Multiply

$$2\partial_{(\mu}E_{\nu)}{}^{a} = T_{\mu\nu}{}^{a} - 2\Omega_{(\mu|e}{}^{a}E_{|\nu)}{}^{e}$$

and

$$2\partial_{(\mu}\Omega_{\nu)a}{}^{b} = R_{\mu\nu a}{}^{b} + 2\Omega_{(\mu|a}{}^{c}\Omega_{|\nu)c}{}^{b}$$

etc, by θ^{μ}

$\Longrightarrow$ Recursions for Vielbein and connection!

Explicitly:

$$E^{(n+1)a}_{\ \ \mu} = -\frac{i}{n+2}E^{(n)\alpha}_{\ \ \mu}(\Gamma^{a}\theta)_{\alpha}, \qquad n \geq 0$$

and

$$E^{(n+1)a}_{\ \ m} = -\frac{i}{n+1}E^{(n)\ \alpha}_{\ \ m}(\Gamma^{a}\theta)_{\alpha}, \qquad n \geq 0$$

More Vielbein

$$E^{(1)\,\alpha}_{\ \ \mu} = 0$$

and

$$\begin{aligned} E^{(n+1)\,\alpha}_{\ \ \mu} &= \frac{i}{(n+1)(n+2)} E^{(n-1)\,\beta}_{\ \ \mu} (D_1^{cdef})_\beta{}^\alpha G^{(0)}_{cdef} \\ &+ \frac{1}{(n+1)(n+2)} \sum_{r=0}^{n-2} E^{(r)\,\beta}_{\ \ \mu} \Big(\frac{1}{n-r-1} F_1^{ef} + \frac{1}{r+2} F_2^{ef} \\ &+ \frac{n+1}{(n-r-1)(r+2)} F_3^{ef} \Big)^\alpha{}_{\beta\gamma} T^{(n-r-2)}{}_{ef}{}^\gamma, \qquad n \geq 1 \end{aligned}$$

where

$$\begin{aligned} (D_1^{cdef})_\beta{}^\alpha &:= \frac{1}{4} (\theta \mathcal{R}_{ab}{}^{cdef})_\beta (\theta \Gamma^{ab})^\alpha + (\theta \Gamma^a)_\beta (\theta \mathcal{T}_a{}^{cdef})^\alpha \\ (F_1^{ef})^\alpha{}_{\beta\gamma} &:= \frac{3}{2} (\theta \Gamma^{ab})^\alpha (\theta \mathcal{R}_{ab}{}^{cdef})_\beta (\theta \Gamma_{cd})_\gamma \\ (F_2^{ef})^\alpha{}_{\beta\gamma} &:= -\frac{1}{4} (\theta \Gamma^{ab})^\alpha (\theta \Gamma^g)_\beta (\theta \mathcal{S}_{gab}{}^{ef})_\gamma \\ (F_3^{ef})^\alpha{}_{\beta\gamma} &:= 6 (\theta \mathcal{T}_a{}^{bcef})^\alpha (\theta \Gamma^a)_\beta (\theta \Gamma_{bc})_\gamma \end{aligned}$$

Yet more Vielbein

$$E^{(1)}{}_m{}^\alpha = \frac{1}{4}(\theta\Gamma^{ab})^\alpha \omega_{mab} - (\theta\mathcal{T}_m{}^{cdef})^\alpha G^{(0)}_{cdef}$$

and

$$\begin{aligned} E^{(n+1)}{}_m{}^\alpha &= \frac{i}{n(n+1)} E^{(n-1)}{}_m{}^\beta (D_1^{cdef})_\beta{}^\alpha G^{(0)}_{cdef} \\ &+ \frac{i}{n(n+1)} T^{(n-1)}{}_{ef}{}^\beta (D_{2m}{}^{ef})_\beta{}^\alpha \\ &+ \frac{1}{n(n+1)} \sum_{r=0}^{n-2} E^{(r)}{}_m{}^\beta \Big(\frac{1}{n-r-1} F_1^{ef} + \frac{1}{r+1} F_2^{ef} \\ &+ \frac{n}{(n-r-1)(r+1)} F_3^{ef}\Big)^\alpha{}_{\beta\gamma} T^{(n-r-2)}{}_{ef}{}^\gamma, \qquad n \geq 1 \end{aligned}$$

where

$$(D_{2a}{}^{bc})_\beta{}^\alpha := -\frac{1}{4}(\theta\mathcal{S}_{aef}{}^{bc})_\beta (\theta\Gamma^{ef})^\alpha + 6(\theta\Gamma_{ef})_\beta (\theta\mathcal{T}_a{}^{bcef})^\alpha$$

The three-form

The Bianchi identity

$$4\partial_{[M}C_{NPQ\}} = G_{MNPQ}$$

is solved by

$$C^{(0)}_{\mu\nu\sigma} = C^{(0)}_{\mu\nu s} = C^{(0)}_{\sigma mn} = 0\ ,$$
$$4\partial_{[m}C^{(0)}_{npq]} = G^{(0)}_{mnpq}$$

and

$$C^{(n+1)}_{\mu\nu\sigma} = \frac{1}{n+4}\,\theta^\lambda G^{(n)}_{\lambda\mu\nu\sigma}$$
$$C^{(n+1)}_{\mu\nu s} = \frac{1}{n+3}\,\theta^\lambda G^{(n)}_{\lambda\mu\nu s}$$
$$C^{(n+1)}_{\sigma mn} = \frac{1}{n+2}\,\theta^\lambda G^{(n)}_{\lambda\sigma mn}$$
$$C^{(n+1)}_{mnp} = \frac{1}{n+1}\,\theta^\lambda G^{(n)}_{\lambda mnp}\ , \qquad n \geq 0$$

Moreover

$$\theta^\lambda G_{\lambda\mu\nu\sigma} = -3iE_{(\mu}{}^aE_\nu{}^bE_{\sigma)}{}^\delta(\Gamma_{ab}\theta)_\delta$$
$$\theta^\lambda G_{\lambda\mu\nu s} = -iE_\mu{}^aE_\nu{}^bE_s{}^\gamma(\Gamma_{ab}\theta)_\gamma - 2iE_s{}^aE_{(\mu}{}^bE_{\nu)}{}^\gamma(\Gamma_{ab}\theta)_\gamma$$
$$\theta^\lambda G_{\lambda\sigma mn} = -iE_m{}^aE_n{}^bE_\sigma{}^\delta(\Gamma_{ab}\theta)_\delta - 2iE_\sigma{}^aE_{[m}{}^bE_{n]}{}^\gamma(\Gamma_{ab}\theta)_\gamma$$
$$\theta^\lambda G_{\lambda mnp} = -3iE_{[m}{}^aE_n{}^bE_{p]}{}^\gamma(\Gamma_{ab}\theta)_\gamma$$

$\Longrightarrow$ the θ expansion of the C-field

Digression: maximally-supersymmetric superspaces

In a bosonic background $T^{(0)}{}_{ab}{}^{\alpha}$ vanishes. Moreover,

$$
\begin{aligned}
T^{(1)}{}_{ab}{}^{\alpha} = e_a{}^m e_b{}^n \{ & \frac{1}{4}(\theta\Gamma^{pq})^{\alpha} R(\omega)_{mnpq} + 2(\theta \mathcal{T}_{[m}{}^{pqrs})^{\alpha}(\mathcal{D}_{n]} G_{pqrs}) \\
& - 2(\theta \mathcal{T}_{[m}{}^{pqrs} \mathcal{T}_{n]}{}^{p'q'r's'})^{\alpha} G_{pqrs} G_{p'q'r's'} \} \\
= e_a{}^m e_b{}^n & \theta^{\beta} (\mathcal{R}^{Tr}_{mn})_{\beta}{}^{\alpha}
\end{aligned}
$$

where $(\mathcal{R}_{mn})^{\alpha}{}_{\beta}$ is the curvature of

$$(\mathbb{D}_m)^{\alpha}{}_{\beta} := (\mathcal{D}_m)^{\alpha}{}_{\beta} - (\mathcal{T}^{Tr}{}_m{}^{pqrs})^{\alpha}{}_{\beta} G_{pqrs}$$

Killing spinors are parallel with respect to $\mathbb{D}$ $\Longrightarrow$

$$(\mathcal{R}_{mn})^{\alpha}{}_{\beta} = 0$$

$T^{(1)}{}_{ab}{}^{\alpha}$ vanishes $\Longrightarrow$ $T_{ab}{}^{\alpha}$ vanishes identically!

We can solve

$$
\begin{aligned}
E_{\mu}{}^{\alpha} &= \delta_{\mu}{}^{\beta}[\mathcal{P}^{-1/2} sinh\sqrt{\mathcal{P}}]_{\beta}{}^{\alpha} \\
E_m{}^{\alpha} &= E^{(1)}{}_m{}^{\beta}[\mathcal{P}^{-1/2} sinh\sqrt{\mathcal{P}}]_{\beta}{}^{\alpha} \\
E_{\mu}{}^{a} &= 2i\delta_{\mu}{}^{\beta}[\mathcal{P}^{-1} sinh^2\frac{\sqrt{\mathcal{P}}}{2}]_{\beta}{}^{\alpha}(\Gamma^a\theta)_{\alpha} \\
E_m{}^{a} &= e_m{}^a + 2iE^{(1)}{}_m{}^{\beta}[\mathcal{P}^{-1} sinh^2\frac{\sqrt{\mathcal{P}}}{2}]_{\beta}{}^{\alpha}(\Gamma^a\theta)_{\alpha}
\end{aligned}
$$

where

$$[\mathcal{P}]_{\alpha}{}^{\beta} := i(D_1^{mnpq})_{\alpha}{}^{\beta} G_{mnpq}$$

Knowledge of the θ-expansion of the superfield $T_{ab}{}^{\alpha}$, the covariant gravitino field-strength, suffices to obtain the θ-expansion of all other superfields, the vielbein in particular.

On the other hand, in an expansion around flat space, the n-th level ($T^{(n)}{}_{ab}{}^{\alpha}$) of the θ-expansion of the gravitino field-strength can be written schematically as

$$T^{(n)} \sim \frac{\mathcal{O}^{\frac{n}{2}}}{n!}\partial\Psi + U^{(n)}, \qquad n = 2k \; ,$$

$$T^{(n)} \sim \frac{\mathcal{O}^{\frac{n-1}{2}}}{n!}(\theta R + \theta\partial G) + U^{(n)}, \qquad n = 2k+1 \; ,$$

where $U^{(n)}$ is a known expression nonlinear in the fields and $\mathcal{O}$ is a (matrix) differential operator quadratic in θ

Schematically, $\mathcal{O} \sim (\theta\Gamma\theta)\partial$. We have denoted by Ψ, R, G, the gravitino, Riemann tensor and four-form field strength of eleven-dimensional supergravity, respectively.

In other words: **Linearly in the number of fields we can obtain expressions which are exact to all orders in θ.**

Moreover, since $U^{(n)}$ is nonlinear, the equations above can be iterated to any order in the number of fields.

Linear expansion

Expand around the flat-space solution

$$
\begin{aligned}
h_m{}^a &:= e_m{}^a - \delta^a_m = 0 \\
\Psi_m{}^\alpha &= 0 \\
C^{(0)}_{mnp} &= 0
\end{aligned}
$$

To linear order

$$
\begin{aligned}
\omega_{nkm} &= \partial_{[k} h_{m]}{}^a \eta_{an} - \partial_{[m} h_{n]}{}^a \eta_{ak} - \partial_{[n} h_{k]}{}^a \eta_{am} + \ldots \\
G^{(0)}_{abcd} &= 4\delta_a{}^m \delta_b{}^n \delta_c{}^p \delta_d{}^q \partial_{[m} C^{(0)}_{npq]} \ldots \\
T^{(0)}{}_{ab}{}^\alpha &= 2\delta_a{}^m \delta_b{}^n \partial_{[m} \Psi_{n]}{}^\alpha + \ldots \\
R^{(0)}_{mnpq} &= \delta_a{}^m \delta_b{}^n \delta_c{}^p \delta_d{}^q R(\omega)_{mnpq} + \ldots
\end{aligned}
$$

and

$$T^{(n)}_{ab} = \frac{1}{n(n-1)} [\mathcal{O}]_{ab}{}^{ef} T^{(n-2)}_{ef} + \ldots$$

where

$$[\mathcal{O}]_{ab}{}^{ef} := i[\mathcal{M}_{[a}{}^{ef}] \delta_{b]}{}^m \partial_m$$

This can be solved:

$$T_{ab} = [cosh\sqrt{\mathcal{O}}]_{ab}{}^{ef} T^{(0)}_{ef} + [\mathcal{O}^{-1/2} sinh\sqrt{\mathcal{O}}]_{ab}{}^{ef} T^{(1)}_{ef}$$

where

$$T^{(1)}_{ef} = \frac{1}{4} (\theta \Gamma^{cd})^\alpha R^{(0)}_{efcd} + 2(\theta \mathcal{T}_{[e}{}^{cdgh})^\alpha \delta_{f]}{}^m \partial_m G^{(0)}_{cdgh}$$

The linear Vielbein

$$E_\mu{}^\alpha = \delta_\mu^\alpha + \Delta E_\mu{}^\alpha$$

where

$$\begin{aligned}\Delta E_\mu{}^\alpha &:= \frac{i}{6}(D_1^{abcd})_\mu{}^\alpha G^{(0)}_{abcd} \\ &+ \sum_{k=0} \frac{1}{2k+4}\Big(\frac{F_1^{ef}}{(2k+3)(2k+1)} + \frac{F_2^{ef}}{2(2k+3)} \\ &\qquad + \frac{F_3^{ef}}{2(2k+1)}\Big)^\alpha{}_{\mu\beta}\{\frac{\mathcal{O}^k}{(2k)!}T^{(0)}\}_{ef}{}^\beta \\ &+ \sum_{k=0} \frac{1}{2k+5}\Big(\frac{F_1^{ef}}{(2k+4)(2k+2)} + \frac{F_2^{ef}}{2(2k+4)} \\ &\qquad + \frac{F_3^{ef}}{2(2k+2)}\Big)^\alpha{}_{\mu\beta}\{\frac{\mathcal{O}^k}{(2k+1)!}T^{(1)}\}_{ef}{}^\beta\end{aligned}$$

and

$$E_m{}^\alpha = \Delta E_m{}^\alpha$$

where

$$\begin{aligned}\Delta E_m{}^\alpha &:= \Psi_m{}^\alpha + \frac{1}{4}(\theta\Gamma^{ab})^\alpha \omega_{mab} - (\theta\mathcal{T}_m{}^{abcd})^\alpha G^{(0)}_{abcd} \\ &+ i\sum_{k=0}\{\frac{\mathcal{O}^k}{(2k+2)!}T^{(0)} + \frac{\mathcal{O}^k}{(2k+3)!}T^{(1)}\}_{ef}{}^\beta (D_{2m}{}^{ef})_\beta{}^\alpha\end{aligned}$$

More of the linear Vielbein

$$E_\mu{}^a = -\frac{i}{2}(\Gamma^a\theta)_\mu + \Delta E_\mu{}^a$$

where

$$\begin{aligned}
\Delta E_\mu{}^a := &\frac{i}{24}(D_1^{bcde}\Gamma^a\theta)_\mu G^{(0)}_{bcde} \\
&-\sum_{k=0} \frac{i(\Gamma^a\theta)_\alpha}{(2k+5)(2k+4)}\Big(\frac{F_1^{ef}}{(2k+3)(2k+1)} + \frac{F_2^{ef}}{2(2k+3)} \\
&\qquad + \frac{F_3^{ef}}{2(2k+1)}\Big)^\alpha{}_{\mu\beta}\{\frac{\mathcal{O}^k}{(2k)!}T^{(0)}\}_{ef}{}^\beta \\
&-\sum_{k=0} \frac{i(\Gamma^a\theta)_\alpha}{(2k+6)(2k+5)}\Big(\frac{F_1^{ef}}{(2k+4)(2k+2)} + \frac{F_2^{ef}}{2(2k+4)} \\
&\qquad + \frac{F_3^{ef}}{2(2k+2)}\Big)^\alpha{}_{\mu\beta}\{\frac{\mathcal{O}^k}{(2k+1)!}T^{(1)}\}_{ef}{}^\beta
\end{aligned}$$

and

$$E_m{}^a = \delta_m{}^a + \Delta E_m{}^a$$

where

$$\begin{aligned}
\Delta E_m{}^a := &h_m{}^a - i(\Psi_m\Gamma^a\theta) - \frac{i}{8}(\theta\Gamma^{aef}\theta)\ \omega_{mef} \\
&+ \frac{i}{2}(\theta\mathcal{T}_m{}^{bcde}\Gamma^a\theta)\ G^{(0)}_{bcde} \\
&+ \sum_{k=0}\{\frac{\mathcal{O}^k}{(2k+3)!}T^{(0)} + \frac{\mathcal{O}^k}{(2k+4)!}T^{(1)}\}_{ef}{}^\beta(D_{2m}{}^{ef}\Gamma^a\theta)_\beta
\end{aligned}$$

The linear C-field

$$C_{\mu\nu\sigma} = \frac{i}{8}(\Gamma^a\theta)_{(\mu}(\Gamma^b\theta)_\nu(\Gamma_{ab}\theta)_{\sigma)} + \Delta C_{\mu\nu\sigma}$$

where

$$\Delta C_{\mu\nu\sigma} := \sum_{n=0}\{\frac{3i}{4(n+6)}(\Gamma^a\theta)_{(\mu}(\Gamma^b\theta)_\nu \Delta E^{(n)}{}_{\sigma)}{}^{\alpha}(\Gamma_{ab}\theta)_\alpha - \frac{3}{n+5}\Delta E^{(n)}{}_{(\mu}{}^{a}(\Gamma^b\theta)_\nu(\Gamma_{ab}\theta)_{\sigma)}\}$$

and

$$C_{s\mu\nu} = \frac{1}{4}(\Gamma^a\theta)_{(\mu}(\Gamma_{ab}\theta)_{\nu)}\delta^b_s + \Delta C_{s\mu\nu}$$

where

$$\begin{aligned}\Delta C_{s\mu\nu} := \sum_{n=0}\{&\frac{i}{4(n+5)}(\Gamma^a\theta)_\mu(\Gamma^b\theta)_\nu \Delta E^{(n)\alpha}_{s}(\Gamma_{ab}\theta)_\alpha \\ &- \frac{1}{n+4}\Delta E^{(n)a}_{s}(\Gamma^b\theta)_{(\mu}(\Gamma_{ab}\theta)_{\nu)} \\ &+ \frac{2i}{n+3}\Delta E^{(n)}{}_{(\mu}{}^{a}(\Gamma_{as}\theta)_{\nu)} \\ &+ \frac{1}{n+4}\Delta E^{(n)}{}_{(\mu}{}^{\alpha}(\Gamma_{sa}\theta)_\alpha(\Gamma^a\theta)_{\nu)}\}\end{aligned}$$

The linear C-field, more of

$$C_{mn\sigma} = -\frac{i}{2}(\Gamma_{ab}\theta)_\sigma \delta^a_m \delta^b_n + \Delta C_{mn\sigma}$$

where

$$\begin{aligned}\Delta C_{mn\sigma} := \sum_{n=0} \{ & \frac{2i}{n+2} \Delta E^{(n)}{}_{[m}{}^{a} (\Gamma_{n]a}\theta)_\sigma \\ & - \frac{i}{n+2} \Delta E^{(n)\alpha}_{\sigma} (\Gamma_{mn}\theta)_\alpha \\ & - \frac{1}{n+3} \Delta E^{(n)}{}_{[m}{}^{\alpha} (\Gamma_{n]a}\theta)_\alpha (\Gamma^a\theta)_\sigma \}\end{aligned}$$

and

$$C_{mnp} = \Delta C_{mnp}$$

where

$$\Delta C_{mnp} := C^{(0)}_{mnp} - \sum_{n=0} \frac{3i}{n+1} \Delta E^{(n)}{}_{[m}{}^{\alpha} (\Gamma_{np]}\theta)_\alpha$$

The supermembrane

The eleven-dimensional supermembrane can be described by a superembedding

$$Z : \Sigma^{(3|0)} \to M^{(11|32)}$$

whith supercoordinates

$$Z^{\underline{M}} := (X^{\underline{m}}, \theta^{\underline{\mu}})$$

World-volume theory:

$$S = \int_{\Sigma} d\sigma^3 \{\sqrt{-g} + f^* C\}$$

where

$$f^* C := \frac{1}{6} \varepsilon^{mnp} \partial_m Z^{\underline{P}} \partial_n Z^{\underline{N}} \partial_p Z^{\underline{M}} C_{\underline{MNP}}$$

and

$$g_{mn} := \left(\partial_m Z^{\underline{M}} E_{\underline{M}}{}^{\underline{a}}\right) \left(\partial_n Z^{\underline{N}} E_{\underline{N}}{}^{\underline{b}}\right) \eta_{\underline{ab}}$$

Linear coupling

To linear order

$$g_{mn} = G_{mn} + \Delta g_{mn}$$

where

$$G_{mn} := \Pi_m{}^{\underline{a}} \Pi_n{}^{\underline{b}} \eta_{\underline{ab}}$$

$$\Delta g_{mn} := 2\Pi_{(m}{}^{\underline{a}} \partial_{n)} Z^{\underline{N}} \Delta E_{\underline{N}}{}^{\underline{b}}\, \eta_{\underline{ab}}$$

and

$$\Pi_m{}^{\underline{a}} := \partial_m X^{\underline{a}} - \frac{i}{2}(\partial_m \theta \Gamma^{\underline{a}} \theta) \; ; \qquad X^{\underline{a}} := X^{\underline{m}} \delta^{\underline{a}}_{\underline{m}}$$

G_{mn} is the Green-Schwarz metric for flat target space.

The determinant is given by

$$\sqrt{-g} = \sqrt{-G}\, (1 + \Delta g_{mn} G^{mn})$$

and the Wess-Zumino term is

$$\begin{aligned} f^* C = f^* \Delta C + \varepsilon^{mnp} \{ & \frac{i}{4} \partial_m X^{\underline{a}} \partial_n X^{\underline{b}} (\partial_p \theta \Gamma_{\underline{ab}} \theta) \\ & + \frac{1}{8} \partial_m X^{\underline{a}} (\partial_n \theta \Gamma^{\underline{b}} \theta)(\partial_p \theta \Gamma_{\underline{ab}} \theta) \\ & - \frac{i}{48} (\partial_m \theta \Gamma^{\underline{a}} \theta)(\partial_n \theta \Gamma^{\underline{b}} \theta)(\partial_p \theta \Gamma_{\underline{ab}} \theta) \} \end{aligned}$$

To summarize:

$$S = S_{flat} + \int_\Sigma d\sigma^3 \{\sqrt{-G} G^{mn} \Delta g_{mn} + f^* \Delta C\}$$

where

$$\begin{aligned} S_{flat} := \int_\Sigma d\sigma^3 \{\sqrt{-G} + \varepsilon^{mnp} \Big[& \frac{i}{4} \partial_m X^{\underline{a}} \partial_n X^{\underline{b}} (\partial_p \theta \Gamma_{\underline{ab}} \theta) \\ & + \frac{1}{8} \partial_m X^{\underline{a}} (\partial_n \theta \Gamma^{\underline{b}} \theta)(\partial_p \theta \Gamma_{\underline{ab}} \theta) \\ & - \frac{i}{48} (\partial_m \theta \Gamma^{\underline{a}} \theta)(\partial_n \theta \Gamma^{\underline{b}} \theta)(\partial_p \theta \Gamma_{\underline{ab}} \theta) \Big] \} \end{aligned}$$

is the action of a supermembrane in flat eleven-dimensional target space

$\Longrightarrow$ Vertex-operators can be read-off!

Conclusions

- All-order results
- Covariant vertex operators
- Generalization to the M5
- M2, M5 instanton contributions
- Berkovits' prescription