NONCOMMUTATIVE GEOMETRY & COHERENT STATES

nc spaces – determined by objects over them (alg. of local observables / fcs, categories of sheaves)

GLOBAL: gluing charts

LOCAL: nc coordinates $[x_i, x_j] \neq 0$

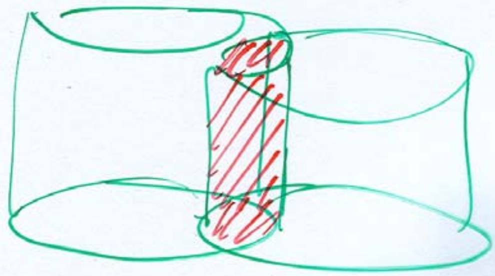
PHYSICS: Hamiltonians given in local coordinates but solutions of eqs. are global e.g. sections of bundles $\Gamma_{hol}\{\}$

dim $\Gamma_{hol}\{\}$ given by flas (RIEMANN-ROCH etc.) on COMPACT SPACES

SINGULARITIES; DIVISORS...

matching & ~~ext~~ending solutions globally needed

• AFFINE vs. NONAFFINE AG



can not glue coordinates

but • bundles

• sheaves

• modules

commutative

↓

classical geometry glue over open sets

NONCOMMUTATIVE TOPOLOGY

$$U \cap V \neq V \cap U \quad !!!$$

LOCALIZATION A FUNCTOR

$$Q_U : R\text{-mod} \to \mathcal{A}_U$$

e.g. $\mathcal{A}_U = R_\mu\text{-mod}$

in R_μ more inverses $R[S_\mu^{-1}]$

$S_\mu \subset R$ e.g. ORE set

$$s_1^{-1} t_1 \cdot s_2^{-1} t_2 = (s'')^{-1} t''$$

no topology $Q_U Q_V M \neq Q_V Q_U M$

also $Q_{U\vee V} M$

$$\begin{tikzcd}
 & Q_U M \arrow[r, "\chi_{U,V}"] \arrow[dr, "\bar\chi_{U,V}"'] & Q_U Q_V M \arrow[d, dashed] \\
M \arrow[ur] \arrow[dr] & & Q_{U\vee V} M \\
 & Q_V M \arrow[r, "\chi_{V,U}"'] \arrow[uur, "\bar\chi_{V,U}"] & Q_V Q_U M \arrow[u]
\end{tikzcd}$$

$$M \longrightarrow \prod_{\lambda\in\Lambda} Q_\lambda M \rightrightarrows \prod_{(\lambda,\mu)\in\Lambda\times\Lambda} Q_{\lambda\lambda'} M$$

FLATNESS

Q_λ faithfully flat

$\Downarrow$

matching exact for moduler & complexes

good class HOMOGENEOUS SPACES

in our focus: (compact) Kähler

+ noncommutative

comm.: alg. group (usually matrices a^i_j)

acting transitively;

quantization line bundle

$\Gamma_{pol} L_\chi$ = Hilbert space

noncomm.: HOPF ALGEBRA (t^i_j)

of coord. fcs. on noncomm. gp. scheme

COACTING

$$\rho : V \to V \otimes H$$

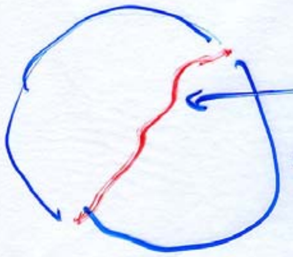
$$(\rho v)(g) = g v$$ ← COMMUTATIVE CASE

COINVARIANTS $\rho V = V \otimes 1$

= inv. fcs = fcs. on coset space

GLOBAL FCS not enough classically

$$\mathcal{O}(SL(2,\mathbb{C})/B) = \mathcal{O}(\mathbb{CP}^1) = \mathbb{C}$$

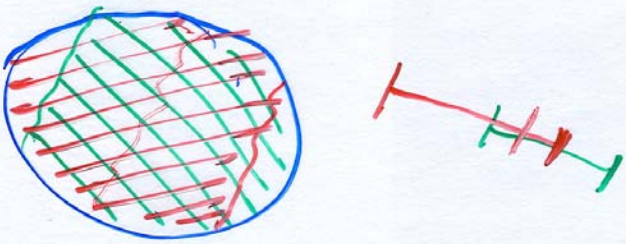


singularity of B-invariant observable

$\underbrace{\text{LOCALIZE}}$, then take coinvariants

INVARIANTLY! localized coinvariants

= PATCHES of NC QUOTIENT



QUANTUM MATRIX GROUPS (t^i_j)

$$\Delta t^i_j = \sum_k t^i_k \otimes t^k_j$$

NC GAUSS $\quad T = w\,U\,A$

w PERM. MATRIX

$$U \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

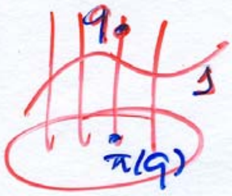
$$A \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \qquad H/I = B$$

$$\pi : H \to B$$

$$b^i_j \longrightarrow a^i_j$$

comodule algebra map

$\left.\begin{matrix} u^i_j \\ a^i_j \end{matrix}\right\}$ in terms of quasideterminants of GEL'FAND, RETAKH



$$s(\pi(q)) = \ell_q(s)\, q$$
$$= \langle e_q | s \rangle\, q$$

$$g\, e_q = \chi^{-1}(b)\, e_{t(gB)q}$$

$$U \subset G^{\mathbb{C}}/B = G/T$$

$$g\, v_0 = \chi^{-1}(b)\, C_w(gB)$$

QUANTUM / NONCOMMUTATIVE

$$\rho_\lambda v_{0\chi} = C_\lambda\, \gamma_\lambda(\chi)$$

$$\gamma : \mathcal{B} \longrightarrow \mathcal{E}_\lambda = \mathcal{E}[S_\lambda^{-1}]$$

V UNITARY IRREP $\quad \Delta_{\mathcal{B}}\chi = \chi \otimes \chi$

$v_{0\chi}$ $\quad (id \otimes \pi)\, \rho\, v_\chi = v_\chi \otimes \chi$

$$\pi : \mathcal{G} \longrightarrow \mathcal{B}$$

$\mathcal{G}$ $*$-HOPF ALGEBRA W/ INVARIANT INTEGRAL

V UNITARY

$$\langle w | z \rangle 1_H = \sum \langle w_{(0)} | z_{(0)} \rangle z_{(1)} w_{(1)}^*$$

$$\langle h \otimes \textstyle\int, \Delta f \rangle = \langle h, 1 \rangle \langle \textstyle\int, f \rangle$$

THM A $\sum \langle w_{(0)} | v \rangle w'_{(0)} \int w_{(1)}^* w_{(1)}'$

is scalar

DEF $d\mu_\lambda(x) = \gamma_\lambda(x) \gamma_\lambda(x)^*$

in $\mathcal{E}_\lambda$

THM A $\rightarrow \int |C_\lambda\rangle d\mu_\lambda(x) \langle C_\lambda|$

is constant

$SU_q(2)$ – much studied $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $SL_q(2)$ (9)

32 papers on $SU_q(2)$ coh. states

NOT COVARIANT C.S.

NOT COVARIANT MEASURE on them

WORK ON CS on quantum groups by Jurčo et.al. (CMP) compatible w/ our picture

other remarkable work: Todorov et.al.

$\mathcal{O}(\mathbb{C}_q^2)$ $\quad xy = qyx$

$\rho(x^r y^s) = (x\otimes a + y\otimes c)^r (x\otimes b + y\otimes d)^s$

$SL_q(2)\,[d^{-1}]\,,\ [b^{-1}]$ $\underline{\text{ORE}}$

B $\begin{pmatrix} \lambda & 0 \\ \xi & \lambda \end{pmatrix}$ $\quad \gamma_d(\lambda) = c - db^{-1}a$

$\gamma_d(\xi) = c$

$\gamma_d(\lambda^{-1}) = d$

$ab = qba,\ ac = qca,\ bd = qdb,\ cd = qdc$

$ad - da = (q - q^{-1})bc,\ \det_q T = ad - qbc = 1$

$a^* = d,\ b^* = -qc,\ c^* = -q^{-1}b,\ d^* = a$

$\{a^k b^r c^s\}_{k>0,\, r,s\geq 0} \cup \{b^r c^s d^t\}_{r,s,t \geq 0}$

$\xi := -qbc \qquad v_i^n = \sqrt{[{n \atop i}]_{q^{-2}}}\, x^i y^{n-i}$

$$\int \xi^r = \frac{1 - q^{-2}}{1 - q^{-2(r+1)}}$$

$$\rho(y^n) = \sum_{i=0}^{n} \left[{n \atop i}\right]_{q^{-2}} q^{-\binom{i}{2}} x^i y^{n-i} \otimes u^i d^n$$

$$C_d := \sum_{i=0}^{n} q^{-\binom{i}{2}} \sqrt{\left[{n \atop i}\right]_{q^{-2}}}\, v_i^n \otimes u^i$$

$$I = q^{-n}[n+1]_q \int_{SU_q(2)} |C\rangle\, d\mu(\lambda) \langle C|$$

$$\int_{SU_q(2)} \xi^i (q^{-2}\xi; q^{-2})_{n-i} = \left[{n \atop i}\right]_{q^{-2}} q^n [n+1]_q^{-1}$$

$$\int_0^1 x^\alpha \frac{(qx;q)_\infty}{(q^\beta x;q)_\infty} d_q x = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$$

RAMANUJAN's q-beta function

$\beta \geq 1$ integer

ratio $(1-x)(1-qx)\cdots(1-q^{\beta-1}x)$

OPEN PROBLEM: covariant minimal uncertainty for CS for Hopf algebras

PROGRAM: Berezin recipe for nc Kähler manifolds & large-N limit for theories over NC spaces