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Matrix Models

Why Mamo?

- show up again and again ...
- Mamo = QFT (ST) / space-time (propagation)

  = pure theory of interactions or of vacua

  arena for landscape
- Mamo are as <u>solvable</u> as possible

What kind of questions we can ask?

Before that: what is MaMo?

$$\int d^{N^2}M \; \exp -\frac{1}{g} \mathrm{Tr}\, W(M)$$

Calculate correlation functions

$$\langle \mathrm{tr}\, M^{k_1} \ldots \mathrm{tr}\, M^{k_n} \rangle$$

eigenvalue model

$$d^{N^2}M = \prod^{N} d\mu_i \; \Delta^2(\mu)$$

$$\Delta^2(\mu) = \prod_{i<j} (\mu_i - \mu_j)^2 = \det{}^2 \mu_i^{j-1}$$

$\to$ orthogonal polynomials

...

Generating functions:

$$Z(t) = \int_C d^{N^2}M \; \exp -\frac{1}{g} \mathrm{Tr}\, W(M) + \frac{1}{g} \mathrm{tr}\; t_k M^k$$

series in t and g

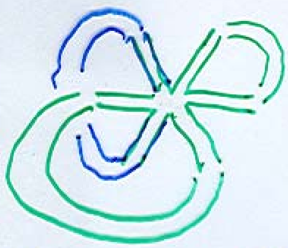
also depend on

N, g, W, C

If $W(z) = z^2$ (Gaussian case)

correlators can be evaluated directly

(catalana numbers)

$\langle \mathrm{tr}\, M^{2k} \rangle =$ 

$$g^{k}\left(a N^{k+1} + b N^{k-1} + \dots\right)$$

$$S_G(z) = \sum \frac{g}{z^{2k+1}} \langle \mathrm{tr}\, M^{2k} \rangle_G =$$

$$= \frac{z - \sqrt{z^2 - 4gN}}{2}$$

$$+ g^2 \frac{4gN}{(z^2 - 4gN)^{5/2}} +$$

$$+ g^4 \frac{21(z^2 + 4gN)}{(z^2 - 4gN)^{11/2}} +$$

$$\dots$$

Every term of genus expansion is well defined.

Entire sum over genera can diverge.

• As we introduce generating function parameter z gets non-trivial

– belongs to the "spectral surface"

hidden entity

$$\rho \sim \left\langle \mathrm{tr} \frac{1}{z-M} \right\rangle$$

$\Rightarrow$ "$\langle M \rangle$" $\in$ Sp. Surf. (curve)

$$y^2 = z^2 - 4gN \quad (\text{h.e. R.S.})$$

• Genus ('t Hooft) expansion

$$g, N \rightarrow g, gN$$

• Other ρ's $\rightarrow$ Quantities on the spectral surface

$\langle \mathrm{tr} M^{k_1} \, \mathrm{tr} M^{k_2} \rangle$

$\rightarrow$ CFT rep.

$\rightarrow$ SFT rep.

$$\rho(z_1, z_2) \sim \left\langle \mathrm{tr} \frac{1}{z_1 - M} \, \mathrm{tr} \frac{1}{z_2 - M} \right\rangle_{\mathrm{conn}}$$

$$\sim \left(\frac{zz' - 4gN}{y(z)\,y(z')} - 1 \right) \frac{dz\, dz'}{(z-z')^2} \approx dd' \log E(z, \overset{*}{z}{}')$$

ρ = correl's

ρ = fields

no pole at $z = z'$

$\rightarrow$ QFT with no pole in the propagator...

Once again: What kind of questions to ask?

- Calculate correlation functions

↓

Generating functions of correlators of given type

→ spectral surfaces

↙ ↘

reps.

CFT SFT

Universal formulas

Correlators = operators on the space of theories

- Phase structure: surprisingly rich

genus expansion

choice of $W(z)$

choice of $f(z)$

- Dualities – equivalencies between differently-looking models

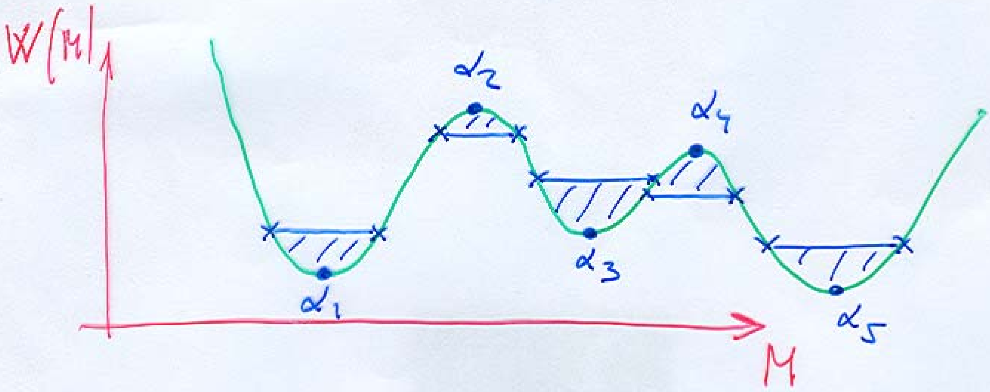
- Partition function = D-module
  SD eqs (loop eqs, Virasoro constraints)
  $$\hat{T} Z = 0 \qquad \hat{T} = \hat{J}_-^2 = 0$$
- CFT rep. $\langle 0 | \hat{T} = 0$
- SFT rep. $S = J^3$
  $$\frac{\partial S}{\partial J} = J_-^2$$

Background independence

- Decomposition formulas
  $$Z = \hat{O} \prod_i \tau_i$$
  model-
  - sewing operator

  Building from elementary blocks
  All phases from one <u>pre</u>-phase

- Special functions
  Integrability
  Group theory $\longrightarrow$ from matrices to tensor algebras



$$Z_W = \hat{O} \prod_{i=1}^{5} Z(\alpha_i)$$

in QM:
instantons

$$= \hat{\tilde{O}} \prod^{10} \tilde{Z}(a)$$

$$\parallel$$

$$\tau(a)$$

merons ?

Kontsevich τ - f

$$\int dX \exp \operatorname{tr}(X^3 + \Lambda X)$$

$$t_k = \frac{1}{k+1/2} \operatorname{tr} \Lambda^{-k-1/2}$$

Beyond Gaussian case:

$$Z(t) \sim \int dM \exp \underset{1/g^2}{\sum} t_k \operatorname{tr} M^k \qquad (*)$$

$$\hat{\mathcal{L}}_n Z = 0 \qquad n \geq -1$$

$$\hat{\mathcal{L}}_n = \sum k t_k \frac{\partial}{\partial t_{k+n}} + \overset{g^2}{\sum_{a+b=n}} \frac{\partial^2}{\partial t_a \partial t_b}$$

$$\hat{\mathcal{L}}(z) = \frac{dz^2}{z^{n+2}} \hat{\mathcal{L}}_n \qquad \delta M = \epsilon M^{n+1}$$

$$\| $$

$$: \hat{J}^2(z) :$$

$$\hat{J}(z) = \frac{1}{2} \sum k t_k z^{k-1} dz + \underset{g^2}{\sum} \frac{dz}{z^{k+1}} \frac{\partial}{\partial t_k}$$

$$J_-^2(z)\, Z = 0 \qquad \frac{\partial \log Z}{\partial t_0} = gN$$

Z - sol'n to this eqn

$(*)$ = integral rep with any int. contour

Phase structure

What kind of solutions do we want?

1) g - dependence

?

$$\hat{\mathcal{L}} = \underset{\kappa t \frac{\partial}{\partial t}}{\hat{\ell}} + g^2 \underset{\frac{\partial^2}{\partial t \partial t}}{\hat{\partial}^2}$$

Taylor series in g^2 — phys. int.

Taylor series in $\frac{1}{g^2}$] too small ?

Laurent series] too big ?

tensor alg. instead of linear alg.

$$Z = \exp \frac{1}{g^2} \mathcal{F}$$

prepotential

Taylor series in g^2

$$\log Z = \sum_{p=0}^{\infty} g^{2p-2} \mathcal{F}_p$$

genus expansion

- a very special class of phases, already physically interesting

2) t - dependence

formal series in positive powers of t's ?

→ no solutions !

smth should go into denominator

⇐ homogeneity prop's

$$t_k \to -T_k + t_k$$

$$-\operatorname{tr} \underset{\overset{\parallel}{T_k M^k}}{W(M)} + \sum t_k \operatorname{tr} M^k$$

formal series in t,

T - in denominator

Different $W \to$ different series/branches/phases

$$S_W(z) \sim \frac{1}{2}\left(W'(z) - \sqrt{W'(z)^2 - 4f(z)}\right) + O(g^2)$$

depends on gN
but not on W

$\deg f = \deg W' - 1$

DV spectral curve

$$y^2 = W'^2 - 4f$$

3) $f(z)$

Phases = branches $\xrightarrow{\text{labeled by}}$ spectral surfaces

$\cup$

Phases with genus expansion ?

(DV phases)

$W_1 \quad W_2 \quad \ldots \quad W$

$f_1 \quad f_2 \quad \ldots \quad f$

Calculate correlation functions

in all phases in a universal way

$S \to \check{S}$ – "check-operators", acting on the space of "base prepotentials" $\mathcal{F}(T)$

$\mathcal{F}(T,t) \longleftarrow \mathcal{F}(T)$

arbitrary

$$\check{S}_W = \frac{1}{2}\left(W' - \sqrt{W'^2 - 4g\sum kT_k \frac{\partial}{\partial T_{k+\ell-2}} z^{\ell}}\right) + O(g^2) =$$

$$= \frac{1}{2}\left(W' - \sqrt{W'^2 - 4g\check{R}(z)}\right) + O(g^2)$$

Matrix as a SFT (String field theory)

$$J^2_-(z) = \frac{\delta S}{\delta J(z)}$$

$$\text{"} \int \bar\partial\phi\,\partial\phi + \int \bar\partial\phi \frac{(\partial\phi)^2}{\Omega_{SW}} \text{"} \simeq S$$

$$\downarrow$$

$$\sum_{a_i} \oint_{a_i} \phi \frac{(\partial\phi)^2}{\Omega_{SW}}$$

→ diagram technique for g's [B. Eynard]

- Ultralocal cubic interaction
- kinetic term = result of shift $\partial\phi \to \partial\phi + \frac{1}{2}\Omega$

as needed in

SFT – ultralocality (causality)
– background independence

$$\Omega_{SW} = y(z)\,dz$$

SW ↑ Seiberg-Witten-Dijkgraaf-Vafa