Gerbes & Branes

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on $\boxed{U_\alpha \cap U_\beta \cap U_\gamma}$

$$d(a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha}) = 0$$

$$\Rightarrow \boxed{a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha} = d\lambda_{\alpha\beta\gamma}}$$

on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

representative of a Čech class

$$2\pi i \mathbb{Z} \ni m_{\alpha\beta\gamma\delta} = \lambda_{\alpha\beta\gamma} + \lambda_{\alpha\gamma\delta} - \lambda_{\beta\gamma\delta} - \lambda_{\alpha\beta\delta}$$

$$[m_{\alpha\beta\gamma\delta}] \in H^3(M, \mathbb{Z})$$

$$g_{\alpha\beta\gamma} = e^{\lambda_{\alpha\beta\gamma}} : U_{\alpha\beta\gamma\delta} \to U(1)$$

$$\boxed{g_{\alpha\beta\gamma}\, g_{\alpha\gamma\delta} = g_{\beta\gamma\delta}\, g_{\alpha\beta\delta}}$$ ← Čech 2-cocycle

$(g_{\alpha\beta\gamma}, a_{\alpha\beta}, B_\alpha)$ ← representative of a Deligne 2-class

gerbe, connection / curving

$\boxed{\text{Abelian 1-gerbe}}$ (with connection an curving)

$H = dB_\alpha$ - curvature (Chern class)

Remark:

Deligne 1-class $(g_{\alpha\beta}, A_\alpha)$

$$g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1 \qquad dA_\alpha = F$$

$$A_\alpha = A_\beta + g_{\alpha\beta}\, dg_{\alpha\beta}^{-1}$$

is a line bundle with connection

What is B-field?

Kapustin, Witten, Freed, Bouwknegt, Mathai, Murray, Carey, Stevenson, Gawedzki....

- one possible answer is

curving of an abelian gerbe

but then

what is a gerbe?

(gerbe = stack of groupoids)

B-field - 2form defined only locally on space-time M

$$M = \bigcup_\alpha \mathcal{U}_\alpha \quad \text{← good covering}$$

on $\boxed{\mathcal{U}_\alpha}$ $\quad B_\alpha$ such that

$\boxed{H = dB_\alpha}$ is globally def.

$$\int_{3\text{-cycle}} H \in 2\pi i \mathbb{Z}$$

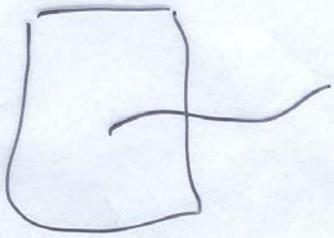
$[H] = h_{\mathbb{R}} \in H^3(M,\mathbb{R})$

$h \in H^3(M,\mathbb{Z})$

on $\boxed{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}$

$$\boxed{B_\alpha - B_\beta = d a_{\alpha\beta}}$$

Q - brane $\quad Q$



$\underline{H|_Q = d\mathcal{F}}$ globally on Q

More precisely $\quad h|_Q = 0 \in H^3(Q,\mathbb{Z})$

on $U_\alpha \cap Q \quad d(\mathcal{F} - B_\alpha) = 0$

$\Rightarrow \boxed{\mathcal{F} = B_\alpha + dA_\alpha}$

$d\mathcal{F} = H$

$\mathcal{F} = B + F$ well defined on Q

on $U_\alpha \cap U_\beta \cap Q \quad B_\alpha - B_\beta = d(A_\alpha - A_\beta) = d(a_{\alpha\beta})$

$$\boxed{A_\alpha - A_\beta = a_{\alpha\beta} + d\Lambda_{\alpha\beta}}$$

on $U_\alpha \cap U_\beta \cap U_\gamma \cap Q$

$$a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha} = d(\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha}) = \\ = d(\lambda_{\alpha\beta\gamma})$$

$$\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha} = \lambda_{\alpha\beta\gamma} \in 2\pi i\mathbb{Z}$$

$\xi_{\alpha\beta} = e^{\Lambda_{\alpha\beta}}$

$$\boxed{g_{\alpha\beta\gamma} = \xi_{\alpha\beta}\xi_{\beta\gamma}\xi_{\gamma\alpha}}$$

$$(g_{\alpha\beta\gamma}, a_{\alpha\beta}, B_\alpha)\big|_Q = D(\xi_{\alpha\beta}, A_\alpha) + (1, 0, \mathcal{F})$$

$\Downarrow$

D-brane Q supports a trivializa-
tion of gerbe related to B

More generally

if $\quad h|_Q \neq 0 \quad$ in $\quad H^3(Q, \mathbb{Z})$

but $\quad n.h = 0 \quad$ on Q

$$\left.\begin{aligned} \xi_{\alpha\beta} &\to \tilde{\xi}_{\alpha\beta} \\ A_\alpha &\to \tilde{A}_\alpha \end{aligned}\right\} \; U(n)\ (u(n)) \text{ valued}$$

Q-stack on D-brane

$$dA_\alpha \to d\tilde{A}_\alpha + \hat{A}_\alpha \wedge \hat{A}_\alpha = \tilde{F}_\alpha$$

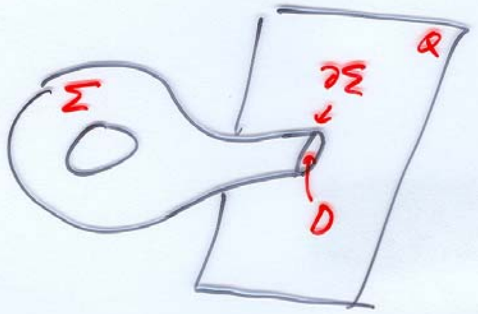
$$\underset{U(1)}{g_{\alpha\beta\gamma}} = \tilde{\xi}_{\alpha\beta}\tilde{\xi}_{\beta\gamma}\tilde{\xi}_{\gamma\alpha} \leftarrow U(n)$$

$$\underset{u(n)}{\tilde{A}_\alpha} = \tilde{\xi}_{\alpha\beta}\tilde{A}_\beta\tilde{\xi}_{\alpha\beta}^{-1} + \hat{\xi}_{\alpha\beta}\, d\hat{\xi}_{\alpha\beta}^{-1} + \overset{u(1)}{a_{\alpha\beta}}$$

$$\mathcal{F} = B_\alpha + \tilde{F}_\alpha$$ – well defined on Q

– U(n)-bundle with connection twisted by gerbe

Why do we need all this ?



Y - 3cycle
$\partial Y = \Sigma \cup D$

we want to make sense out of the partition function

$$e^{\int_\Sigma B} \; \mathrm{Tr}\, \mathrm{hol}_{\partial\Sigma} \tilde{A}$$

not a true connection

B is not globally defined

$$e^{\int_Y H} \; e^{\int_D B + dA} \; \mathrm{Tr}\, \mathrm{hol}_{\partial\Sigma} (\tilde{A} - A)$$

- this makes perfect sense

on D - gerbe is trivial (restriction to 2-dim D of gerbe coming from $h \in H^3(M, \mathbb{Z})$)

So we have A $u(1)$ values

$B + dA$ is globally defined $\tilde{\xi}_{\alpha\beta} \xi_{\alpha\beta}^{-1}$
and $\tilde{A} - A$ is a true connection

Remarks: 1. Superstrings

$$h|_Q = W_3(NQ)$$

Freed Witten global anomaly

(Pfaffian of worldsheet Dirac) in partition function

$2W_3 = 0$

W_3 - 2 torsion

2. If $h|_Q - W_3(NQ)$

is nontorsion

$$U(n) \longrightarrow U(\mathcal{H})$$

Bouwknegt Mathai (twisted K)

3. $h|_Q - W_3(NQ)$ is torsion

Abelian gerbe (with connection, curving) $\Leftrightarrow$ obstruction to lift a $PU(n)$ bundle (with connection)

$$1 \longrightarrow U(1) \longrightarrow U(n) \longrightarrow PU(n) \rightarrow 1$$

nontorsion $\Leftrightarrow$ obstruction to lift a $PU(\mathcal{H})$ bundle

$$1 \longrightarrow U(1) \longrightarrow U(\mathcal{H}) \rightarrow PU(\mathcal{H}) \rightarrow 1$$

(torsion case - reduction of structure group)

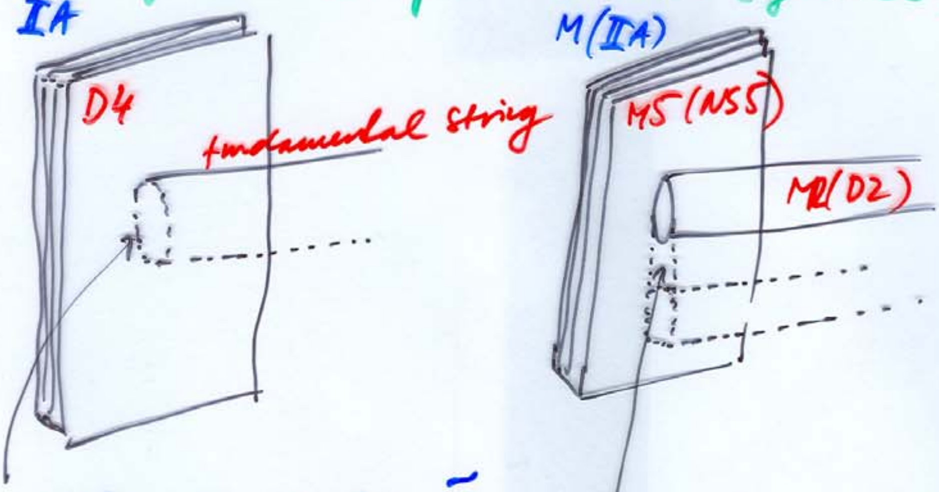
dim $M = 10$ we can replace this by

$$1 \rightarrow U(1) \longrightarrow \widetilde{\Omega E_8} \rightarrow \Omega E_8 \rightarrow 1$$

higher forms?

e.g. G 4-form of M-theory

higher Deligne classes (gerbes)



1-cycle couples to $\tilde{A}$
(connection on a $\widetilde{\Omega E_8}$
bundle twisted by
abelian 1-gerbe)

2-cycle should couple
to a nonabelian
2-form
(living on a nonabelian
$\widetilde{\Omega E_8}$-gerbe twisted
by an abelian 2-form)

$\widetilde{\Omega E_8}$ central extension of based E_8 loops replaces naively Freed-Witten

$$\mathcal{W}_4(NM5) = G|_{M5}$$

- More precisely <u>global anomaly</u> has 2 parts

topological ← Witten $\theta \in H^4_{Torsion}(M5, \mathbb{Z})$

$\theta \neq 0$ partition function of isolated M5 vanishes

and a part depending on background fields Freed, Diaconescu and Moore

$\vartheta(g, G)$

partition function $\sim \Psi_{Bulk} \Psi_{M5}$

gauge invariance $\vartheta = 0$ $\qquad \theta = 0$

$\vartheta = \theta$ $\qquad \Psi_{Bulk} \cdot \Psi_{M5}$

is still gauge invariant

If $\vartheta - \Theta \neq 0$ in $H^4(M5, \mathbb{Z})$

we use the following fact

$$H^4(M5, \mathbb{Z}) \sim E_8 \text{ bundles on } M5$$

$$\begin{array}{ccc} E_8 \to P & & \\ \downarrow & \leftrightarrow & \vartheta - \Theta = p_1(P) \\ M5 & & \end{array}$$

based paths

$$1 \longrightarrow \Omega E_8 \longrightarrow PE_8 \longrightarrow E_8 \to 1$$

there is an obstruction to lift
E_8 bundle $P \to M5$ to a PE_8
– nonabelian ΩE_8 gerbe $\mathcal{G}$

$$1 \to U(1) \to \widetilde{\Omega} E_8 \to \Omega E_8 \to 1$$

Again, there is an obstruction to
lift ΩE_8 gerbe $\mathcal{G}$ to $\widetilde{\Omega} E_8$
– abelian 2-gerbe $\sim \vartheta - \Theta \in H^4(M5, \mathbb{Z})$

What we have a the end is
a nonabelian $\widetilde{\Omega} E_8$ gerbe twisted
by an abelian 2-gerbe
(+ connections
curvings ...) Its curving is the
nonabelian 2-form gauge
potential