Geometry of the Fuzzy Donut

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   first order (and beyond)
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1. Motivation

- Uncertainty relations + general relativity imply lower limit on coordinate measurements
- Such a lower limit (in principle) provides a natural UV cutt-off i.e. a regulator in quantum field theory
- in the spirit of Einstein's heuristic methods
- Tools : loop gravity
  strings, branes
  noncommutative geometry

Classical mechanics

singular limit: change of structure, Hilbert space

Quantum mechanic

General relativity

singular limit: notion of manifold lost

Noncommutative gravity

commutative limit: also singular?

Elements for generalization

- **Differential structure**

  it is possible to define differential noncommutative geometry, i.e. forms and vector fields on algebras

- **Symplectic structure**

  given in quantum mechanics by $[x^i, p_j] = i\hbar\delta^i_j$; momenta are related to the derivatives

Noncommutative space = algebra generated by the **coordinates** x^i + relations ; for example
$[x^i, x^j] = i\hbar J^{ij}(x)$

J^{ij} = const : canonical structure, or flat space

Vector fields are derivations, i.e. linear mappings X : algebra $\to$ algebra which obey the Leibniz rule,

$$X(gh) = Xg \cdot h + g \cdot Xh$$

- generalized derivatives : deformed Leibniz!

- vector fields do not form a left module, i.e. if X is a derivation, hX in general is not

Forms : **1-forms** are linear maps ω : vectors $\to$ algebra which have bimodule structure, i.e. along with ω, $h\omega$ and ωh are 1-forms

- p-forms - generalization of wedge product

- differential : Leibniz + $d^2 = 0$

2. Differential calculus

NC generalization of Cartan's moving frame formalism:

Frame is defined by a set of n 1-forms θ^i. It can also be defined by dual vector fields e_i : $\theta^i(e_j) = \delta^i{}_j$

Madore's version: in a sense, minimal
- all derivations are inner, $e_i h = [p_i, h]$
- momenta p_i generate the algebra as well as the coordinates x^i
- the center of the algebra is trivial

Advantages
- better notion of dimension
- commutative limit
- structure constrained

Drawbacks
- structure too constrained ?

given a frame θ^i, the **differential** is defined by

$$dh(e_i) = e_i h \quad , \quad \text{or} \qquad dh = [p_i, h]\,\theta^i$$

- Various **constraints** appear. For example

$$h\theta^i = \theta^i h \qquad \text{for all elements } h \text{ of the algebra}$$

- Or: from $d(h\theta^i - \theta^i h) = 0$, and $d^2 = 0$ a quadratic relation among the momenta can be obtained

$$2P^{kl}{}_{ij}\, p_k p_l - F^k{}_{ij}\, p_k - K_{ij} = 0$$

- Also, we always assume associativity, i.e. the Jacobi identities hold

The simplest **example** is the flat space:

$$[x^i, x^j] = i\hbar J^{ij} = \text{const}$$

$$\theta^i = dx^i \quad , \qquad [x^j, dx^i] = 0$$

$$dx^j dx^i = -dx^i dx^j$$

$$p_j = \frac{1}{i\hbar} J^{-1}_{jk}\, x^k$$

2-dim space with a Killing vector

$$\theta^0 = f(x)\, dt \quad , \quad \theta^1 = dx$$

Define $[t, x] = i\hbar J^{01}$

Calculus is given by

$$dx \cdot x = x\, dx \qquad dt \cdot x = x\, dt$$
$$dx \cdot t = t \cdot dx \qquad dt \cdot t = (t + i\hbar F)\, dt$$

with $F = J^{01} f' \cdot f^{-1}$.

Also, $dJ^{01} = 0$ ($J^{01} = \mathrm{const}$), and

$$(\theta^1)^2 = 0$$
$$\theta^0 \theta^1 = -\theta^1 \theta^0$$
$$(\theta^0)^2 = \frac{1}{2} i\hbar f F' \theta^0 \theta^1 .$$

However! if derivatives are inner and momenta generate the algebra, we have further restrictions. They are

$$p_0 = -\frac{1}{i\hbar} \int f^{-1} dx \quad , \quad p_1 = -\frac{1}{i\hbar} t$$

but also the equation

$$-i\hbar \frac{dp_0}{dx} = 1 - i\hbar b\, p_0 + a (i\hbar p_0)^2 .$$

The last equation fixes in fact $f(x)$, i.e. the allowed frames. It has 3 solutions:

1) $f(x) = 1$, flat space (zero curvature)

2) $f(x) = e^{cx}$, AdS (constant curvature)

3) $f(x) = \cosh^2 \beta x = \frac{1}{2}(1 + \cosh 2\beta x)$,

fuzzy donut or torus

as, after the Wick rotation $u = 2i\beta x$, $v = t$, it becomes

$$\theta^0 = \frac{1}{2}(1+\cos u)dv \ , \ \theta^1 = \frac{1}{2i\beta} du$$

which can be embedded in $\mathbb{R}^3$ as a singular torus.

$$p_0 = -\frac{1}{i k\beta} \tanh \beta x$$

$$\tilde{\Omega}^0{}_1 = -2\beta^2 (1 + \tanh \beta x)\, \theta^0 \theta^1$$

nonconstant curvature

Exterior product, using the frame, can be defined as

$$\theta^i \theta^j = P^{ij}{}_{kl}\, \theta^k \otimes \theta^l$$

where $P^{ij}{}_{kl}$ are constants. $P^{ij}{}_{kl}$ is a projector; in the usual case, $P^{ij}{}_{kl} = \frac{1}{2}(\delta^i{}_k \delta^j{}_l - \delta^i{}_l \delta^j{}_k)$ - antisymmetrization

Further, to define the linear connections one needs a 'flip', or change of order in the tensor product

$$\sigma(\theta^i \otimes \theta^j) = S^{ij}{}_{kl}\, \theta^k \otimes \theta^l$$

Usually: $S^{ij}{}_{kl} = \delta^i{}_l \delta^j{}_k$; gives symmetrization $(1+\sigma)$

Then, the covariant derivative is defined as

$$D\xi = \sigma(\xi \otimes \theta) - \theta \otimes \xi \qquad \text{with } \theta = p_i \theta^i$$

and connection as

$$D\theta^i = -\omega^i{}_j \otimes \theta^j .$$

Metric, as a map $g(\theta^i \otimes \theta^j) = g^{ij}$, etc.

The formalism fully developed, once $P^{ij}{}_{kl}$ and $S^{ij}{}_{kl}$ are given.

3. Differential geometry

Solve for $P^{ij}{}_{kl}$ and $S^{ij}{}_{kl}$ for the fuzzy donut, assuming some additional requirements like hermiticity, etc. Find, perhaps, metric (real and symmetric), and connection (torsion-free and metric-compatible).
In general, difficult.

One can try a semiclassical approximation, i.e. expansion in orders of $\hbar$:

$$P^{ij}{}_{kl} = \frac{1}{2}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) + i\hbar Q^{ij}{}_{kl}$$

$$S^{ij}{}_{kl} = \delta^i_l \delta^j_k + i\hbar T^{ij}{}_{kl}$$

$$g^{ij} = \eta^{ij} + i\hbar h^{ij} \quad , \quad etc.$$

Differential calculus fixes $Q^{ij}{}_{kl}$, at least in the first order. From

$$(\theta^0)^2 = 2i\hbar\theta^0\theta^1$$

we get

$$Q^{10}{}_{00} = -Q^{01}{}_{00} = Q^{00}{}_{01} = -Q^{00}{}_{10} = 1 .$$

3. Differential geometry

The zero-th order connection gives $T^{ij}{}_{kl}$:

$$T^{00}{}_{01} = T^{10}{}_{00} = -4 \, ,$$

whereas the other conditions give the metric

$$g^{ij} = \begin{pmatrix} -1 & 2ik \\ 0 & 1 \end{pmatrix} .$$

It is real and symmetric. To this order connection (metric compatible and torsion-free) and curvature are

$$\omega^0{}_1 = \omega^1{}_0 = -4ik\, p_0\, \theta^0 = F\theta^0$$

$$\Omega^0{}_1 = -(F' + F^2)\,\theta^0\theta^1$$

$$\Omega^0{}_0 = \Omega^1{}_1 = 2ik\, F^2\, \theta^0\theta^1$$

It is possible to go to the second order
The corrected metric is real and symmetric; the connection is real and torsion-free, but not metric-compatible

3. Differential geometry

Nonperturbatively : conditions can be written in the matrix form: if we write $P^{ij}{}_{kl}$ and $S^{ij}{}_{kl}$ as 4×4 matrices, g^{ij} as a column and denote the 'flat' values as P_0, S_0, g_0, then the constraints are

- projector constraints:

$$P^2 = P \quad , \quad P^* \hat{P} = \hat{P} \qquad \text{with } \hat{P} = S_0 P$$

- twist constraints

$$\hat{S}^* \hat{S} = 1 \quad , \quad (S+1)P = 0$$

- metric constraints

$$\hat{g}^* = Sg$$

$$Pg = 0 \quad \text{or} \quad Sg = cg$$

- connection metric-compatible

$$\omega^i{}_{kl}\, g^{lj} + \omega^j{}_{ln}\, S^{il}{}_{km}\, g^{mn} = 0$$

4. Conclusions

2d :

- in the special case of dependence on 1 coordinate, geometry is almost fixed
- more general dependence : difficult to obtain exact results

4d :

- many more examples : more coordinates allow more flexibility
- still , it is unclear whether the correct classical limit can be obtained

generalizations :

- higher dimensions + dimensional reduction
- outer derivations