Cubic Algebras and Generalized Statistics

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Yang -Mills Algebra

- the associative algebra behind the Yang-Mills equations in $d+1$-dimensional (pseudo)Euclidean space with pseudo metric $g_{\mu\nu}$

$$g^{\lambda\mu}[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0 \qquad (1)$$

$\nabla_\mu = \partial_\mu + A_\mu$, $\mu = 0, 1, ..., d$ is the gauge covariant derivative

- by definition this associative algebra is a universal enveloping algebra, generated by the $d+1$ generators ∇_μ with the $d+1$ cubic relations (1), the Yang-Mills equations

interest in YM algebra in connection with string theory
N.Nekrasov - Lectures on open strings and non-commutative gauge theories hep-th/0203109

introduced by Green as a generalization of the Bose-Fermi alternative.

DEFINITION 1 *The parafermi algebra* $\mathfrak{pF}(n)$ *(parabose algebra* $\mathfrak{pB}(n)$*) is an associative algebra generated by the creation* a^{+i} *and annihilation* a_i^- *operators for* $i = 1, \ldots, n$ *subject to the relations*

$$\begin{aligned} [[a^{+i}, a_j^-]_\mp, a^{+k}] &= 2\delta_j^k a^{+i} \\ [[a^{+i}, a^{+j}]_\mp, a^{+k}] &= 0 \\ [[a^{+i}, a_j^-]_\mp, a_k^-] &= -2\delta_k^i a_j^- \\ [[a_i^-, a_j^-]_\mp, a_k^-] &= 0 \end{aligned} \quad (1)$$

The upper (lower) sign refers to the parafermi algebra $\mathfrak{pF}(n)$ (parabose algebra $\mathfrak{pB}(n)$).

Superalgebraic point of view:

$$\begin{aligned} [\![[\![a^{+i}, a_j^-]\!], a^{+k}]\!] &= 2\delta_j^k a^{+i} \\ [\![[\![a^{+i}, a^{+j}]\!], a^{+k}]\!] &= 0 \\ [\![[\![a^{+i}, a_j^-]\!], a_k^-]\!] &= -2\delta_k^i a_j^- \\ [\![[\![a_i^-, a_j^-]\!], a_k^-]\!] &= 0 \end{aligned} \qquad (2)$$

where $[\![a, b]\!] = ab - (-1)^{deg(a)deg(b)} ba$ and $deg(x) \in \{\bar{0}, \bar{1}\}$ is the $\mathbb{Z}_2$ degree of x. Then for the parafermi $\mathfrak{pF}(n)$ (parabose $\mathfrak{pB}(n)$) case all the generators are even (odd)

$$\text{deg}(a^{+i}) = deg(a_j^-) = \bar{0},$$

$$(\text{deg}(a^{+i}) = deg(a_j^-) = \bar{1}),$$

The parastatistics algebras admit an antilinear antiinvolution $*$, $(ab)^* = b^* a^*$

$$(a^{+i})^* = a_i^- \quad (a_i^-)^* = a^{+i}$$

referred to as conjugation.

The parafermi algebra $\mathfrak{pF}(n)$ is isomorphic to UEA of the orthogonal algebra $so(2n+1)$, the parabose algebra $\mathfrak{pB}(n)$ is isomorphic to UEA of the orthosymplectic algebra $osp(1|2n)$

$$\mathfrak{pF}(n) \simeq U(so(2n+1))$$

$$\mathfrak{pB}(n) \simeq U(osp(1|2n))$$

Thus the trilinear relations (1) provide an alternative set of relations for the algebras $so(2n+1)$ and $osp(1|2n)$ in terms of paraoscillators.

Deformed Parastatistics Algebras

The idea of quantization of the parastatistics algebras is to "quantize" the isomorphisms, i.e., to deform the trilinear relations (1) so that the arising deformed parafermi $\mathfrak{pF}_q(n)$ and parabose $\mathfrak{pB}_q(n)$ algebras are isomorphic to the QUEAs

$$\mathfrak{pF}_q(n) \simeq U_q(so(2n+1))$$

$$\mathfrak{pB}_q \simeq U_q(osp(1|2n))$$

.

and then continue the algebraic isomorphism to a Hopf morphism which endows the deformed parastatistics with a natural Hopf structure.

QUEA $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ in the Chevalley-Serre form - same Cartan matrix

$(C_{ij})_{i,j=1,\ldots,n}$ with entries

$$C_{ij} = \alpha_j(H_i) = (\alpha_i^\vee, \alpha_j)$$

$$\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -2 & 2 \end{pmatrix} \qquad (3)$$

Symmetrized Cartan matrix $(a_{ij})_{i,j=1\ldots n}$

$$a_{ij} = d_i C_{ij} = (\alpha_i, \alpha_j) \qquad d_i = \frac{(\alpha_i, \alpha_i)}{2}$$

which in the cases under consideration is

$$a_{ij} = 2\delta_{ij} - \delta_{in} - \delta_{i+1j} - \delta_{ij+1} \qquad d_i = 1 - \frac{1}{2}\delta_{in} \qquad (4)$$

Let $H_i, E_{\pm i}$ be the Chevalley basis of $so(2n+1)$ or $osp(1|2n)$

$$H^{\alpha_i} = H_i, \qquad E^{\pm\alpha_i} = E_{\pm i} \qquad 1 \leq i \leq n. \tag{5}$$

The Lie superalgebra $osp(1|2n)$ has a grading induced by $deg(H_i) = \bar{0}$ and

$$deg(E_{\pm i}) = \bar{0} \quad 1 \leq i \leq n-1 \qquad deg(E_{\pm n}) = \bar{1} \tag{6}$$

All generators of the Lie algebra $so(2n+1)$ are even.

The QUE algebras $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ are associative algebras generated by $q^{\pm H_i}$ and $E_{\pm i}$ subject to the relations

$$
\begin{array}{ll}
q^{H_i}q^{H_j} = q^{H_j}q^{H_i} & i,j \leq n \\
q^{H_i}E_{\pm j}q^{-H_i} = q^{\pm a_{ij}}E_{\pm j} & i,j \leq n \\
[2][E_i, E_{-j}] = \delta_{i,j}[2H_i] & i \leq n-1 \\
[\![E_n, E_{-n}]\!] = [H_n] & \\
[E_{\pm i}, E_{\pm j}] = 0 & |i-j| \geq 2 \\
[E_{\pm i}, [E_{\pm i}, E_{\pm(i+1)}]_q]_{q^{-1}} = 0 & i \leq n-1 \\
[E_{\pm(i+1)}, [E_{\pm(i+1)}, E_{\pm i}]_q]_{q^{-1}} = 0 & i \leq n-2 \\
[[\![[E_{\pm(n-1)}, E_{\pm n}]_{q^{-1}}, E_{\pm n}]\!], E_{\pm n}]_q = 0 &
\end{array}
\tag{7}
$$

where

$$
[x] := \frac{q^{\frac{x}{2}} - q^{-\frac{x}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \qquad (= [x]_{q^{\frac{1}{2}}}).
$$

Change of basis by using the subset of short roots ε_i' related to the simple roots by

$$
\alpha_i = \varepsilon_i - \varepsilon_{i+1} \qquad 1 \leq i \leq n-1, \qquad \alpha_n = \varepsilon_n \tag{8}
$$

Corresponding change of basis on the Cartan subalgebra

$$H_i = h_i - h_{i+1} \qquad 1 \leq i \leq n-1, \qquad\qquad H_n = h_n. \tag{9}$$

By construction $q^{h_i} q^{h_j} = q^{h_j} q^{h_i}$.

The ladder operators $E^{+\varepsilon_i}$ and $E^{-\varepsilon_i}$ related to the roots ε_i are a^{+i} and a_j^- and therefore the inverse change $\varepsilon_i = \sum_{k=i}^{n} \alpha_k$ implies

$$\begin{aligned} a^{+i} &= [E_i, [E_{i+1}, \ldots [E_{n-1}, E_n]_{q^{-1}} \ldots]_{q^{-1}}]_{q^{-1}} \\ a_i^- &= [[\ldots [E_{-n}, E_{-n+1}]_q \ldots, E_{-(i+1)}]_q, E_{-i}]_q \end{aligned} \tag{10}$$

On the other hand the Chevalley generators are expressed as

$$\begin{aligned} E_i &= \frac{1}{[2]} q^{-h_{i+1}} [\![a^{+i}, a_{i+1}^-]\!] \\ E_{-i} &= \frac{1}{[2]} [\![a^{+(i+1)}, a_i^-]\!] q^{h_{i+1}} \qquad i < n \\ E_n &= a^{+n} \qquad\qquad E_{-n} = a_n^- \end{aligned} \tag{11}$$

One has

$$q^{h_i} a^{+j} q^{-h_i} = q^{\delta_{ij}} a^{+j} \qquad q^{h_i} a_j^- q^{-h_i} = q^{-\delta_{ij}} a_j^- \quad (12)$$

The graded commutator of opposite ladder operators

$$[\![a^{+i}, a_i^-]\!] = [2h_i] \qquad (13)$$

defines the partial hamiltonian $\mathcal{H}_i$ (of the i-th paraoscillator)

$$\mathcal{H}_i = \frac{1}{[2]}[\![a^{+i}, a_i^-]\!] = \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \qquad (14)$$

The full hamiltonian $\mathcal{H}$ is the sum over all paraoscillators $\mathcal{H} = \sum_{i=1}^n \mathcal{H}_i$.

Antiinvolution $*$ on the new generators

$$(a^{+i})^* = a_i^- \qquad (a_i^-)^* = a^{+i} \qquad (q^{\pm h_i})^* = q^{\mp h_i} \qquad (15)$$

and also $(q)^* = q^{-1}$, (q on the unit circle). (and for the Chevalley basis $(E_{\pm i})^* = E_{\mp i}$, $H_i^* = H_i$)

Hence $*$ is an antiinvolution on the whole QUEA.

THEOREM 1 *The quantum parafermi* $\mathfrak{pF}_q(n)$ *(parabose* $\mathfrak{pB}_q(n)$*) algebra is the associative (super)algebra generated by the even Cartan generators* q^{h_i} *for* $i = 1, \ldots, n$ *and the even (odd) raising* a^{+i} *and lowering* a_i^- *generators subject to the relations*

$$[\![[\![a^{+i}, a_j^-]\!], a^{+k}]\!]_{q^{-\delta_{ik}\sigma(j,k)}} = [2]\delta_j^k a^{+i} q^{\sigma(i,j)h_j}$$

$$+(q\text{-}q^{-1})\theta(i,j;k)a^{+i}[\![a^{+k}, a_j^-]\!] \quad (16)$$

$$\begin{aligned}
[\![[\![a^{+i_1}, a^{+i_3}]\!], a^{+i_2}]\!]_{q^2} + q[\![[\![a^{+i_1}, a^{+i_2}]\!], a^{+i_3}]\!] &= 0\\
[\![[\![a^{+i_1}, a^{+i_2}]\!], a^{+i_2}]\!]_q &= 0\\
[\![a^{+i_2}, [\![a^{+i_1}, a^{+i_3}]\!]]\!]_{q^2} + q[\![a^{+i_1}, [\![a^{+i_2}, a^{+i_3}]\!]]\!] &= 0\\
[\![a^{+i_2}, [\![a^{+i_2}, a^{+i_3}]\!]]\!]_q &= 0
\end{aligned} \quad (17)$$

$i_1 < i_2 < i_3$

as well as their conjugated

$$[\![[\![a^{+i}, a_j^-]\!], a_k^-]\!]_{q^{-\delta_{jk}\sigma(i,k)}} = -[2]\delta_k^i a_j^- q^{-\sigma(i,j)h_i}$$

$$-(q-q^{-1})\theta(j,i;k)[\![a^{+i}, a_k^-]\!]a_j^- \quad (18)$$

$$\begin{aligned}
[\![[\![a_{i_1}^-, a_{i_3}^-]\!], a_{i_2}^-]\!]_{q^2} + q[\![[\![a_{i_1}^-, a_{i_2}^-]\!], a_{i_3}^-]\!] &= 0 \quad i_1 < i_2 < i_3 \\
[\![[\![a_{i_1}^-, a_{i_2}^-]\!], a_{i_2}^-]\!]_q &= 0 \quad i_1 < i_2 \\
[\![a_{i_2}^-, [\![a_{i_1}^-, a_{i_3}^-]\!]]\!]_{q^2} + q[\![a_{i_1}^-, [\![a_{i_2}^-, a_{i_3}^-]\!]]\!] &= 0 \quad i_1 < i_2 < i_3 \\
[\![a_{i_2}^-, [\![a_{i_2}^-, a_{i_3}^-]\!]]\!]_q &= 0 \quad i_2 < i_3
\end{aligned} \quad (19)$$

where $\sigma(i,j) = \epsilon_{ij} + \delta_{ij}$ (* *or* $\sigma(i,j) = \epsilon_{ij} - \delta_{ij}$ *and* $\theta(i,j;k) = \frac{1}{2}\epsilon_{ij}\epsilon_{ijk}(\epsilon_{jk} - \epsilon_{ik})$(†.

The inhomogeneous relations are related to the adjoint action of a deformed linear algebra. The homogeneous relations describe an ideal which is a $U_q(gl(n))$-module, a deformation of a Schur module $E^{(2,1)}$.

*) Levi-Chevita symbol $\epsilon_{ij} = 1$ for $i < j$

†)The function $\theta(i,j;k) = -\theta(j,i;k)$ is 0 ; or 1 and -1 for $i < k < j$ and $i > k > j$, respectively.

Hopf structure on parastatistics algebras

The QUE algebras $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ endowed with the Drinfeld-Jimbo coalgebraic structure

$$
\begin{array}{rclrcl}
\Delta H_i & = & H_i \otimes 1 + 1 \otimes H_i & S(H_i) & = & -H_i \\
\Delta E_i & = & E_i \otimes 1 + q^{H_i} \otimes E_i & S(E_i) & = & -q^{-H_i} E_i \\
\Delta E_{-i} & = & E_{-i} \otimes q^{-H_i} + 1 \otimes E_{-i} & S(E_{-i}) & = & -E_{-i} q^{H_i}
\end{array}
\qquad (20)
$$

$$\epsilon(H_i) = \epsilon(E_i) = \epsilon(E_{-i}) = 0$$

become Hopf algebra and Hopf superalgebra, respectively. (Superalgebras have a graded Hopf structure with antipode which is a *graded* antihomomorphism

$$S(ab) = (-1)^{deg(a)deg(b)} S(b)S(a). \qquad (21)$$

The conjugation $*$ (15) for $|q| = 1$ is a coalgebraic antihomomorphism, $(\Delta x)^* = \sum (x_{(1)} \otimes x_{(2)})^* = \sum x^*_{(2)} \otimes x^*_{(1)}$ and $S(x^*) = S(x)^*$ for $x \in U_q$.

The isomorphisms to the QUEA induce Hopf structure on the deformed parastatistics algebras.

THEOREM 2 *The deformed parafermionic algebra $\mathfrak{pF}_q(n)$, the deformed parabosonic algebra $\mathfrak{pB}_q(n)$ is a Hopf algebra, a Hopf superalgebra, respectively when endowed with*

(i) a coproduct Δ defined on the generators by

$$\Delta q^{\pm h_i} = q^{\pm h_i} \otimes q^{\pm h_i} \qquad (22)$$

$$\Delta a^{+i} = a^{+i} \otimes 1 + q^{h_i} \otimes a^{+i} + \omega \sum_{i<j\leq n} [\![a^{+i}, a_j^-]\!] \otimes a_j^+$$

$$\Delta a_i^- = a_i^- \otimes q^{-h_i} + 1 \otimes a_i^- - \omega \sum_{i<j\leq n} a_j^- \otimes [\![a^{+j}, a_i^-]\!]$$

(ii) a counit ϵ defined on the generators by

$$\epsilon(q^{\pm h_i}) = 1 \qquad \epsilon(a^{i+}) = \epsilon(a_i^-) = 0 \quad (25)$$

(iii) an antipode S (graded for $\mathfrak{pB}_q(n)$) defined on the generators by

$$S(q^{\pm h_i}) = q^{\mp h_i} \qquad (26)$$

$$S(a^{+i}) = -q^{-h_i}a^{+i} - \sum_{s=1}^{n-i}(-\omega)^s \sum_{i<j_1<\ldots<j_s\leq n} W^{+}{}^{i}_{j_1} W^{+}{}^{j_1}_{j_2}$$

$$S(a_i^-) = -a_i^- q^{h_i} - \sum_{s=1}^{n-i}(\omega)^s \sum_{n\geq j_s>\ldots>j_1>i} a_{j_s}^- q^{h_{j_s}} W^{-}$$

where $W^{+}{}^{i}_{j} = q^{-h_i}[\![a^{+i}, a_j^-]\!]$, $W^{-}{}^{j}_{i} = [\![a^{+j}, a_i^-]\!]q^{h_i}$ *and* $\omega = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$.

Recall some basic tools for QUEA $U_q(g)$ of a simple Lie algebra g:

The QUEA $U_q(g)$ is generated by the elements of an upper-triangular and a lower triangular matrices $L^{(+)}$ and $L^{(-)}$

$$R^{(+)}L_1^{(\pm)}L_2^{(\pm)} = L_2^{(\pm)}L_1^{(\pm)}R^{(+)}$$

$$R^{(+)}L_1^{(+)}L_2^{(-)} = L_2^{(-)}L_1^{(+)}R^{(+)}$$

where $L_1^{(\pm)} = 1 \otimes L^{(\pm)}$, $L_2^{(\pm)} = L^{(\pm)} \otimes 1$ and $R^{(+)} = PRP$ is the corresponding R-matrix for $U_q(g)$.

The Hopf structure on the elements of $L^{(+)}$ and $L^{(-)}$ compatible with the Drinfeld structure (defined on the Chevalley basis) is given by the coproduct $\Delta L^{\pm}$, the counit $\epsilon(L^{(\pm)})$

$$\Delta L_k^{i(\pm)} = \sum_j L_j^{i(\pm)} \otimes L_k^{j(\pm)} \qquad \epsilon(L_k^{i(\pm)}) = \delta_k^i \tag{29}$$

and the antipode $S(L^{(\pm)})$

$$\sum_j L_j^{i(\pm)} S(L_k^{j(\pm)}) = \delta_k^i = \sum_j S(L_j^{i(\pm)}) L_k^{j(\pm)}. \tag{30}$$

Consider the QUEA $U_q(so(2n+1))$. The matrices $L^{(+)}$ and $L^{(-)}$ are $(2n+1) \times (2n+1)$

matrices with elements in $U_q(so(2n+1))$. The corner $L^{i(+)}_j$, $1 \leq i,j \leq n+1$ of the matrix $L^{(+)}$ is very simple when expressed in terms of the generators a^{+i} and a^-_j

$$\left(L^{i(+)}_j\right)_{1\leq i,j\leq n+1} =$$

$$\begin{pmatrix} q^{h_1} & \omega[a^{+1},a^-_2] & \omega[a^{+1},a^-_3] & \dots & \omega[a^{+1},a^-_n] & ca^{+1} \\ 0 & q^{h_2} & \omega[a^{+2},a^-_3] & \dots & \omega[a^{+2},a^-_n] & ca^{+2} \\ 0 & 0 & q^{h_3} & \dots & \omega[a^{+3},a^-_n] & ca^{+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q^{h_n} & ca^{+n} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$\omega = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. The coefficient $c = q^{-\frac{1}{2}}(q - q^{-1})$.

A similar result holds for $U_q(osp(1|2n))$ but instead of commutators one has to take anticommutators.

Summarizing the formulae for QUEA of Lie (super)algebras of the series $B(n)$ and $B(0|n)$, the left $n \times n$ minor of the upper-triangular matrix $L^{(+)}$ reads

$$\begin{aligned} L_i^{i(+)} &= q^{h_i} && \text{for } 1 \le i \le n \\ L_j^{i(+)} &= \omega [\![a^{+i}, a_j^-]\!] && \text{for } 1 \le i < j \le n \end{aligned} \tag{32}$$

The conjugation $*$ (15) exchanges the upper-triangular matrix $L^{(+)}$ and the lower-triangular matrix $L^{(-)}$

$$(L_j^{i(+)})^* = L_i^{j(-)}. \tag{33}$$

Coproduct: $\Delta L_{n+1}^{i(+)} = \sum_{j=1}^{2n+1} L_j^{i+} \otimes L_{n+1}^{j+} =$

$= L_{n+1}^{i(+)} \otimes 1 + \sum_j L_j^{i(+)} \otimes L_{n+1}^{j(+)}$

$L_{n+1}^{i(+)} = c\, a^{i(+)}$

$$\Delta a^{+i} = a^{+i} \otimes 1 + \sum_{i \le j \le n} L_j^{i(+)} \otimes a^{+j}$$

Deformed

$pF_q(n)$ paraoperators $pB_q(n)$

Coproduct Δ

$$\Delta a^{+i} = a^{+i} \otimes 1 + q^{h_i} \otimes a^{+i} + \omega \sum_{i<j\le n} [\![a^{+i}, a_j^-]\!] \otimes a^{+j}$$

$$\Delta a_i^- = a_i^- \otimes q^{-h_i} + 1 \otimes a_i^- - \omega \sum_{i<j\le n} a_j^- \otimes [\![a^{+j}, a_i^-]\!]$$

Antipode S

$$S(a^{+i}) = -q^{-h_i} a^{+i} - \sum_{s=1}^{n-i} (-\omega)^s \sum_{i<j_1<\dots<j_s\le n} W^{+i}_{j_1} W^{+j_1}_{j_2} \dots W^{+j_{s-1}}_{j_s} q^{-h_{j_s}} a^{+j_s}$$

$$S(a_i^-) = -a_i^- q^{h_i} - \sum_{s=1}^{n-i} (\omega)^s \sum a_{j_s}^- q^{h_{j_s}} W^{-j_s}_{j_{s-1}} \dots W^{-j_2}_{j_1} W^{-j_1}_{i}$$

$$W^{+i}_{j} = q^{-h_i} [\![a^{+i}, a_j^-]\!], \quad W^{-j}_{i} = [\![a^{+j}, a_i^-]\!] q^{h_i},$$

$$\omega = q^{1/2} - q^{-1/2}$$

The oscillator representations

The unitary representations π_p of the parastatistics algebras $\mathfrak{pB}(n)$ and $\mathfrak{pF}(n)$ with unique vacuum state are indexed by a non-negative integer p . The representation π_p is the lowest weight representation with a unique vacuum state $|0\rangle$ annihilated by all a_i^- and labelled by the *order of parastatistics* p

$$\pi_p(a_i^-)|0\rangle = 0 \qquad\qquad \pi_p(a_i^-)\pi_p(a^{+j})|0\rangle = p\delta_i^j|0\rangle\,. \tag{34}$$

The *vacuum representation* (the trivial one with $p = 0$) is given by the counit

$$\pi_0(x)|0\rangle = \epsilon(x)|0\rangle \qquad\qquad x \in \mathfrak{pB}(n), \mathfrak{pF}(n). \tag{35}$$

In the representation π_p of the nondeformed parastatistics algebras the hamiltonian $h_i = \frac{1}{2}[a^{+i}, a_i^-]_\mp$ and the number operator $N_i = a^{+i}a_i^-$ of the i-th paraoscillator are related by

$$h_i = N_i \mp \frac{p}{2} \tag{36}$$

where the upper (lower) sign is for parafermions (parabosons). The constant $\mp\frac{p}{2}$ plays the role of the energy of the vacuum. In the representation π_p of the deformed parastatistics algebras the quantum analogue of the relation holds

$$[a^{+i}, a_i^-]_\mp = [2]\mathcal{H}_i = [2h_i] = [2N_i \mp p]$$

which implies the deformed analogue of the π_p defining condition

$$a_i^-(p)a^{+j}(p)|0\rangle^{(p)} = [p]\delta_i^j|0\rangle^{(p)} . \qquad (37)$$

The constant $\mp[p]/[2]$ plays the role of energy of the vacuum

$$\mathcal{H}_i|0\rangle^{(p)} = \mp\frac{[p]}{[2]}|0\rangle^{(p)} .$$

The algebra of the q-deformed bosonic oscillators $\mathfrak{B}_q(n)$ arises as a particular representation

π of parabosonic order $p = 1$ of the $\mathfrak{pB}_q(n)$

$$\begin{array}{rcl} \underline{a}_i^- \underline{a}^{+i} - q\underline{a}^{+i}\underline{a}_i^- & = & q^{-\underline{N}_i} \\ \underline{a}^{+i}\underline{a}^{+j} - q^{\epsilon_{ij}}\underline{a}^{+j}\underline{a}^{+i} & = & 0 \\ \underline{a}_i^- \underline{a}_j^- - q^{\epsilon_{ij}}\underline{a}_j^- \underline{a}_i^- & = & 0 \end{array} \qquad \begin{array}{r} \underline{a}_i^- \underline{a}^{+i} - q^{-1}\underline{a}^{+i} \\ \underline{a}^{+i}\underline{a}_j^- - q^{\epsilon_{ji}}\underline{a}_j^- \underline{a} \\ \underline{a}_i^- \underline{a}^{+j} - q^{\epsilon_{ji}}\underline{a}^{+j} \end{array} \tag{38}$$

where $\pi(x) = \underline{x}$ and $\underline{N}_i = \underline{h}_i - \frac{1}{2}$.

The algebra of the q-deformed fermionic oscillators $\mathfrak{F}_q(n)$ is the $p = 1$ representation of the parafermionic algebra $\mathfrak{pF}_q(n)$

$$\begin{array}{rcl} \underline{a}_i^- \underline{a}^{+i} + q\underline{a}^{+i}\underline{a}_i^- & = & q^{\underline{N}_i} \\ \underline{a}^{+i}\underline{a}^{+j} + q^{\epsilon_{ji}}\underline{a}^{+j}\underline{a}^{+i} & = & 0 \\ \underline{a}_i^- \underline{a}_j^- + q^{\epsilon_{ji}}\underline{a}_j^- \underline{a}_i^- & = & 0 \\ (\underline{a}^{+i})^2 & = & 0 \end{array} \qquad \begin{array}{r} \underline{a}_i^- \underline{a}^{+i} + q^{-1}\underline{a}^{+i}\underline{a} \\ \underline{a}^{+i}\underline{a}_j^- + q^{\epsilon_{ij}}\underline{a}_j^- \underline{a} \\ \underline{a}_i^- \underline{a}^{+j} + q^{\epsilon_{ij}}\underline{a}^{+j}\underline{a} \\ (\underline{a}_i^- \end{array} \tag{39}$$

where $\underline{N}_i = \underline{h}_i + \frac{1}{2}$.

Green Ansatz

The Green ansatz states - The parafermi (parabose) oscillators a^{+i} and a_i^- can be represented as sums of p fermi (bose) oscillators

$$\pi_p(a^{+i}) = \sum_{r=1}^{p} a^{+i}_{(r)} \qquad \pi_p(a_i^-) = \sum_{r=1}^{p} a^-_{i(r)} \tag{40}$$

satisfying quadratic commutation relations of the same type (i.e., fermi for parafermi and bose for parabose) for equal indices (r)

$$[a^-_{i(r)}, a^{+k}_{(r)}]_\pm = \delta_i^k, \quad [a^-_{i(r)}, a^-_{k(r)}]_\pm = [a^{+i}_{(r)}, a^{+k}_{(r)}]_\pm = 0 \tag{41}$$

and of the opposite type for the different indices

$$[a^-_{i(r)}, a^-_{k(s)}]_\mp = [a^{+i}_{(r)}, a^{+k}_{(s)}]_\mp = [a^-_{i(r)}, a^{+k}_{(s)}]_\mp = 0, r \neq s \tag{42}$$

The upper (lower) signs stay for the parafermi (parabose) case.

The coproduct endows the tensor product of $\mathcal{A}$-modules of the Hopf algebra $\mathcal{A}$ with the structure of an $\mathcal{A}$-module. Thus one can use the coproduct for constructing a representation out of simple ones. The simplest representations of the parastatistics algebras are the oscillator representations π (with $p = 1$). A parastatistics algebra representation of arbitrary order arises through the iterated coproduct.

Let us denote the (p-fold) iteration of the coproduct by

$$\Delta^{(0)} = id, \Delta^{(1)} = \Delta, \Delta^{(p)} = (\underbrace{\Delta \otimes 1 \ldots \otimes 1}_{p-1}) \circ \Delta^{(p-1)} \tag{43}$$

and π denotes the projection from the (deformed) parafermi and parabose algebra onto the (deformed) fermionic $\mathfrak{F}$ ($\mathfrak{F}_q$) and bosonic $\mathfrak{B}$ ($\mathfrak{B}_q$)

Fock representation, respectively. Then we have

$$\begin{aligned} \pi_p(a^{+i}) \equiv \pi^{\otimes p} \circ \Delta^{(p)}(a^{+i}) \ &:= \ \sum_{r=1}^{p} a^{+i}_{(r)} \\ \pi_p(a_i^-) \equiv \pi^{\otimes p} \circ \Delta^{(p)}(a_i^- \ &:= \ \sum_{r=1}^{p} a^-_{i(r)} \end{aligned} \tag{44}$$

Consistency of the vacuum condition with the deformed Green ansatz. The vacuum state $|0\rangle^{(p)}$ of the representation π_p is to be identified with the tensor power of the oscillator ($p = 1$) vacuum, $|0\rangle^{(p)} = |0\rangle^{\otimes p}$. Evaluating the iterated graded commutator

$$\Delta^{(p)}[\![a^{+i}, a_i^-]\!] = \frac{(q^{h_i})^{\otimes p} - (q^{-h_i})^{\otimes p}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \tag{45}$$

on the vacuum state $|0\rangle^{\otimes p}$ in the oscillator representations $\pi^{\otimes p}$ we get the defining condition of the deformed π_p

$$\mp\pi^{\otimes p}\circ\Delta^{(p)}[\![a^{+i}, a_i^-]\!]|0\rangle^{(p)} = \pi_p(a_i^-)\pi_p(a^{+i})|0\rangle^{(p)} = [$$

$$= [p]|0\rangle^{(p)}$$

since $\pi(q^{h_i}) \equiv q^{N_i \mp \frac{1}{2}}$, which proves the consistency.

The Green components $a^{+i}_{(r)}$ and $a^{-}_{i(r)}$ in a $\mathfrak{pF}_q(n)$ or $\mathfrak{pB}_q(n)$ representation π_p of parastatistics of order p will be chosen to be

$$
\begin{aligned}
a^{+i}_{(r)} &= \Delta^{(r-1)} \otimes 1 \otimes \Delta^{(p-r)} \left(\sum_{k=1}^{n} L_k^{i(+)} \otimes a^{+k} \otimes 1 \right) \\
a^{-}_{i(r)} &= \Delta^{(r-1)} \otimes 1 \otimes \Delta^{(p-r)} \left(\sum_{k=1}^{n} 1 \otimes a^{-}_{k} \otimes L_i^{k(-)} \right)
\end{aligned}
\tag{46}
$$

Note that the conjugation $*$ acts as reflection on the Green indices (r)

$$(a^{+i}_{(r)})^* = a^{-}_{i(r^*)} \quad (a^{-}_{i(r)})^* = a^{+i}_{(r^*)} \quad r^* = p - r + 1.$$

The exchange relations of the Green components of the deformed Green ansatz close quadratic algebras .

For different Green indices:

($[x,y]_{\pm q} = xy \pm qyx$) ($r > s$)

$$
\begin{aligned}
i<j \quad [a^{+i}_{(r)}, a^{+j}_{(s)}]_{\mp} &= \mp(q-q^{-1})a^{+j}_{(r)}a^{+i}_{(s)} \qquad 0 = [a^{-}_{i(r)}, a^{-}_{j(s)}] \\
i>j \quad [a^{-}_{i(r)}, a^{-}_{j(s)}]_{\mp} &= \pm(q-q^{-1})a^{-}_{j(r)}a^{-}_{i(s)} \qquad 0 = [a^{+i}_{(r)}, a^{+}_{(s)}]
\end{aligned}
\tag{47}
$$

$$
[a^{+i}_{(r)}, a^{+i}_{(s)}]_{\mp q} = 0 \qquad r>s \qquad [a^{-}_{i(r)}, a^{-}_{i(s)}]_{\mp q^{-1}} = 0 \tag{48}
$$

$$
[a^{-}_{i(r)}, a^{+j}_{(s)}]_{\mp} = 0 \qquad \text{for} \qquad r \neq s \tag{49}
$$

For equal Green indices:

$$
\begin{aligned}
[a^{+i}_{(r)}, a^{+j}_{(r)}]_{\pm q^{\mp\epsilon_{ij}}} &= 0 \quad , \qquad 0 = [a^{-}_{i(r)}, a^{-}_{j(r)}]_{\pm q} \\
[a^{-}_{i(r)}, a^{+j}_{(r)}]_{\pm q^{\mp 1}} &= q^{\mp\frac{1}{2}} Q^{j(-)}_{i(r)}, \qquad q^{\pm 1/2} Q^{j+}_{i(r)} = [a^{-}_{i(r)}, a^{+j}_{(r)}]_{\pm q^{\pm}}
\end{aligned}
\tag{50}
$$

where the operators $Q^{j(+)}_{i(r)}$ and $Q^{j(-)}_{i(r)}$ are quadratic in the Green components

$$q^{\mp\frac{1}{2}}Q^{j(-)}_{i(r)} = (q-q^{-1})\sum_{s=1}^{r-1} q^{\mp(r-s)} a^{+j}_{(s)} a^{-}_{i(s)} \quad i>j$$
$$q^{\mp\frac{1}{2}}Q^{j(-)}_{i(r)} = -(q-q^{-1})\sum_{s=r}^{p} q^{\mp(r-s)} a^{+j}_{(s)} a^{-}_{i(s)} \quad i<j \tag{51}$$

$$q^{\pm\frac{1}{2}}Q^{i(+)}_{i(r)} = q^{\mp(r-\frac{p}{2}-\frac{1}{2})}(q^{N_i})^{\otimes p} - (q-q^{-1})\sum_{s=r+1}^{p} Q$$
$$q^{\mp\frac{1}{2}}Q^{i(-)}_{i(r)} = q^{\mp(r-\frac{p}{2}-\frac{1}{2})}(q^{-N_i})^{\otimes p} + (q-q^{-1})\sum_{s=1}^{r-1} Q \tag{52}$$

$$Q = q^{\mp(r-s)} a^{+i}_{(s)} a^{-i}_{i(s)}$$

The upper (lower) signs are for the parafermi (parabose) case. For the parafermi algebra $\mathfrak{pF}_q(n)$ one has in addition

$$(a^{+i}_{(r)})^2 = 0 \quad (a^{-}_{i(s)})^2 = 0 \quad \text{for } \mathfrak{pF}_q(n) \tag{53}$$