Exercises on the course
"Introduction into Cosmic Structure Formation"

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1. a) Consider a linear functional of a Gaussian random field $\delta(\mathbf{q})$,

$$\tilde{\delta}(\mathbf{q}) = \int d^3\mathbf{q}'\, T(\mathbf{q},\mathbf{q}')\, \delta(\mathbf{q}')\ ,$$

   where T is some kernel. Prove that $\tilde{\delta}$ is also a Gaussian random field.
   b) Consider random distribution of point masses

$$\rho(\mathbf{x}) = \sum_\alpha m\ \delta^{(D)}(\mathbf{x}-\mathbf{x}_\alpha). \tag{1}$$

   Here $\delta^{(D)}$ is the Dirac δ-function, and $\mathbf{x}_\alpha$ are positions chosen randomly within a certain volume. Find $P(k)$. Is the field (1) Gaussian ?

2. Show by an explicit solution of the Maxwell equations in an expanding universe that the frequency of an electromagnetic wave redshifts as $1/a(\tau)$.

3. Derive the hydrodynamic equations,

$$\frac{\partial\rho}{\partial\tau} + \partial_i(\rho v^i) = 0\ , \tag{2a}$$

$$\frac{\partial v^i}{\partial\tau} + v^j\partial_j v^i = -\partial_i\phi - \frac{1}{\rho}\partial_j\sigma^{ij}\ , \tag{2b}$$

$$\Delta\phi = 4\pi G\rho\ , \tag{2c}$$

   from the Vlasov–Poisson system of equations. Find the expression for σ^{ij} in terms of the phase-space distribution function.

4. Take the distribution function in the form,

$$f(\tau;\mathbf{x},\mathbf{V}) = \rho(\tau,\mathbf{x})\ \delta^{(D)}(\mathbf{V}-\mathbf{v}(\tau,\mathbf{x}))\ ,$$

   which corresponds to an ensemble of particles with zero velocity dispersion. Show that this Ansatz satisfies the Vlasov–Poisson system, provided that ρ and $\mathbf{v}$ satisfy Eqs. (2) with $\sigma^{ij}=0$.

5. Find the behavior of the cold dark matter density contrast $\delta_{CDM} \equiv \frac{\delta\rho_{CDM}}{\bar{\rho}_{CDM}}$ and the Newtonian potential $\delta\phi$ at the Λ-dominated stage. Assume $\rho_\Lambda \gg \rho_{CDM}$.

6. In the non-relativistic limit the dynamics of a massive self-gravitating scalar field is described by the Schrödinger–Poisson system,

$$i\dot{\psi} + \frac{\Delta^2 \psi}{2m} - m\,\phi\,\psi = 0\;, \tag{3a}$$

$$\Delta\phi = 4\pi G m^2 |\psi|^2\;. \tag{3b}$$

a) Introduce new variables ρ, v^i using the relations,

$$\psi = \frac{\sqrt{\rho}}{m}\mathrm{e}^{i\theta}\;, \qquad v^i = \frac{1}{m}\partial_i\theta\;.$$

Show that in terms of these variables the system (3) takes the form of the Euler–Poisson equations (2) with some stress tensor σ^{ij}. Find the expression for σ^{ij}.
b) Derive the equations for small perturbations of the ψ-density. Determine the Jeans length.
c) Assuming that ψ constitutes dark matter and $m \sim 10^{-22}\,\mathrm{eV}$, discuss the modifications of the power spectrum and the halo mass function compared to the case of CDM made of heavy particles.

7. Consider a CDM with a Yukawa interaction between dark matter particles mediated by a light scalar field with mass $m_\chi \gg H$. The potential between two particles in the non-relativistic limit reads,

$$V(r) = -\frac{Gm_1m_2}{r}(1 + \alpha \mathrm{e}^{-m_\chi r})\;.$$

Find the equations describing spherical collapse in this system. Analyze the qualitative properties of the solution. Assuming α is small, find the leading-order corrections to the Newtonian spherical collapse.

8. Find the evolution of the dark matter density contrast during radiation domination stage for modes shorter than the free-streaming length,

$$k > \frac{aH}{\bar{u}}\;,$$

where $\bar{u}$ is the mean velocity of the unperturbed dark matter phase-space distribution.

9. Estimate the velocity dispersion of weakly interacting massive particles (WIMPs) at the moment of matter-radiation equality. Take the values for the WIMP mass and decoupling temperature, $m = 100\,\mathrm{GeV}$, $T_{dec} = 10\,\mathrm{MeV}$.

10. Obtain a lower bound on the mass of fermionic dark matter from existences of halos with dark matter density in their central regions $\rho \sim 5 \times 10^7 M_\odot/(\mathrm{kpc})^3$. Take the size of the central region to be $l \sim 1$ kpc.